Nonlinear vibrations in plates and gongs

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The sounds from certain types of gongs show a frequency shift, up or down, as the vibration decays. In order to understand this nonlinear behavior, we have studied the vibrations of a variety of flat and curved plates under varying amounts of radial tension or compression. Flat plates show a nonlinearity of a hardening type (falling frequency), whereas most curved plates show softening behavior (rising frequency). The degree of the nonlinearity appears to be sufficient to explain the observed frequency shift in the gongs studied.

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INTRODUCTION

Considerable interest has been shown in the nonlinear vibrations of flat plates,[1,2] curved plates,[3] and conical and spherical shells.[4,5] Since the equations describing these vibrations cannot be solved exactly, various techniques have been employed for obtaining approximate solutions. Theoretical investigations have been more commonly reported than experiments.

Our own interest in the problem arises from the acoustical behavior of certain gongs, such as the gongs used in Chinese opera orchestras. These latter gongs are characterized by a marked pitch glide, the frequency changing by as much as 20% as the sound dies away after striking.[6,7]

A vibrating system is described as *hardening* if its vibrational frequency increases with increasing amplitude and *softening*, if its frequency decreases. Flat plates exhibit hardening, whereas conical and spherical shells generally exhibit softening. Curved plates can be of either type, depending upon their shape.

It is the purpose of this paper to describe the principal modes of vibration of flat and curved plates as well as certain types of gongs. Vibrational frequencies have been measured both when the plates are driven sinusoidally and when struck percussively with a mallet. On the basis of these data it is possible to explain the interesting pitch glide (upward or downward) in certain gongs.

Although the vibrations of plates and shells at large and small amplitudes have been discussed extensively in the published literature, most of the discussions are directed toward particular geometries of interest to the authors. Furthermore the notations used vary widely. Thus it is worthwhile to review briefly in the following sections the vibrational theory for plates and shallow spherical shells.

I. SMALL-AMPLITUDE VIBRATIONS OF PLATES

A. Flat circular plates

The well-known differential equation describing flexural motion of a flat plate can be written in the form

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = 0, \qquad (1)$$

where w is the lateral deflection, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the plate, E is Young's modulus, ν is Poisson's ratio, ρ is density, and h is plate thickness.

Solutions to this equation for circularly symmetric plates have the form[8,9]

$$w(r,\theta) = [AJ_m(kr) + BI_m(kr)](\cos m\theta + \alpha), \qquad (2)$$

where $J_m(kr)$ and $I_m(kr)$ are Bessel functions, and A, B, α, and k are constants determined by the boundary conditions and the nature of the excitation.

If the plate is clamped at its edge $r = a$, then the requirements that $w = 0$ and $\partial w/\partial r = 0$ lead to the equations $AJ_m(ka) + BI_m(ka) = 0$ and $AJ'_m(ka) + BI'_m(ka) = 0$. The values of k which satisfy these equations define the normal modes. The allowed values are labeled k_{mn}, where m gives the number of nodal diameters and n the number of nodal circles. Values of k_{mn} are tabulated in various sources.[9,10]

If a radial tension T is applied to the plate, an additional term $-T\nabla^2 w$ appears in Eq. (1). The modal frequency increases when the plate is under tension (positive T) and decreases when a radial compression is applied (negative T).[11] Relative frequencies for the first three modes are plotted in Fig. 1 based on data on Ref. 11. In a real plate the frequency will not fall to zero as shown in Fig. 1, because the plate buckles under compression and the linear theory fails as the critical load for buckling is approached.

B. Shallow spherical shells

In addition to the flexural modes of vibration of a flat plate, discussed in the previous section, a curved shell has many longitudinal, torsional, and thickness-shear modes. Fortunately, the lowest modes are mainly flexural, and the other modes couple only weakly to them under the conditions of interest to us. Thus a simple theory of transverse vibrations is quite accurate for treating the lowest few modes of the thin shells of slight curvature of interest in this study.[12,13]

Equation (1) may be used to describe transverse vibrations of a shallow spherical shell by adding a term $\nabla^2 F/R$, where R is the radius of curvature of the shell, and F is Airy's stress function. A second equation $\nabla^4 F - hE\nabla^2 w/R = 0$ completes the problem.[12,13] For a clamped-edge shell whose apex height H (above the edge plane) is equal to the shell thickness, the fundamental mode (at small amplitude) is

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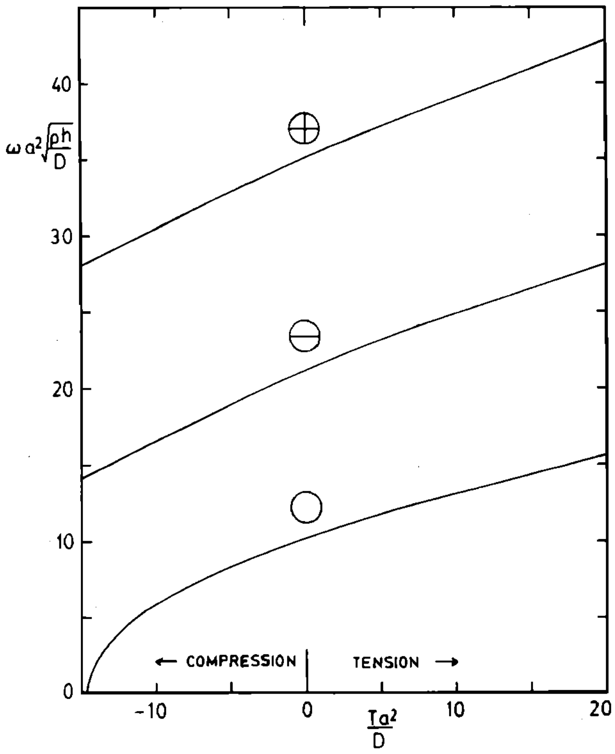

FIG. 1. Theoretical effect of radial tension and compression on the lowest vibrational frequencies of a flat circular plate (from data in Ref. 11). ω is 2π times frequency, a is the radius, ρ is density, h is thickness, D is stiffness, and T is tension.

found to have a frequency about 40% above that of a flat plate with the same thickness and radius.[14] Other modal frequencies for a shell with a free edge are given in Ref. 13 and for a clamped edge in Ref. 14.

A spherical shell also differs from a flat plate in that there are five possible edge conditions to be considered (free, fixed, clamped, hinged, and simply supported), whereas in a flat plate there are only three (free, clamped, and simply supported). The clamped and simply supported conditions allow the edge of a shell to move tangentially, so that inextensional modes can occur.[5]

II. NONLINEAR VIBRATIONS OF PLATES

As the amplitude of vibration increases, the linear theory used to describe small-amplitude vibrations becomes inaccurate. Most nonlinear equations used to describe large-amplitude vibrations cannot be integrated explicitly, and various methods are used to obtain approximate solutions.

A. Flat circular plates

The term which must be added to Eq. (1) to extend to large-amplitude vibrations has the form $-N\nabla^2 w$, where N is the sum of the radial and transverse stress resultants. Physically, this represents an amplitude-dependent membrane-type restoring force due to stretching the plate on the outside of the bulge, compressing it on the inside. When the amplitude equals the plate thickness the frequency increases by about 16%, increasing to 35% at twice the thickness.[15]

B. Shallow spherical shell

In a shallow spherical shell the nonlinearity can be either of the hardening or softening type, depending upon the curvature. In a shell with a fixed edge, softening occurs when the apex height H exceeds the thickness. When H equals twice the thickness and the amplitude equals the thickness, for example, the frequency of vibration is 10% less than its small-amplitude value.[5]

The softening effect at moderately large amplitude offsets some of the increase in vibrational frequency with curvature which was discussed in Sec. I B. Again using as an example a shell with fixed edge having H equal to twice the thickness, at small amplitude the fundamental frequency is approximately twice that of a flat plate, but the ratio diminishes to 1.6 when the amplitude is equal to the thickness.[5]

C. Shells of other shapes

Most of the theoretical and experimental investigations of shallow shell vibrations have dealt with spherical, cylindrical, or conical shells, all of which show a softening type of nonlinearity at moderately large amplitude.

A shell with two independent radii of curvature R_x and R_y can show either hardening or softening behavior, however. For most ratios of R_x/R_y, shells with a rectangular edge, for example, show an initial softening followed by a hardening at very large amplitude. The hyperbolic paraboloid ($R_x/R_y = -1$), however, shows the same type of hardening behavior as a flat plate.[3]

III. EXPERIMENTAL STUDIES

A. Mechanical admittance and modal analysis

A Bruel and Kjaer type 8001 impedance head and type 4810 vibration exciter were used to measure the mechanical admittance of various plates and gongs as a function of driving frequency. The driving point force was kept constant by use of a GenRad 1569 level controller, and the accelerometer output was amplified and integrated with a Bruel and Kjaer type 2651 charge amplifier connected through the tracking filter of a GenRad 1900A analyzer to a GenRad 1521B chart recorder. Several driving points were used for each plate, including the center.

We attempted to determine the modal configuration for as many plate resonances as possible. Normally this was accomplished by moving a small microphone in the nearfield of the radiated sound. The plate-to-microphone spacing was kept as small as possible, and the nodal lines were mapped by noting the change in phase when a node was passed. The principal modes thus determined were in good agreement with the holographic interferograms and Chladni patterns of vibrations in Chinese gongs previously reported.[6] At the higher frequencies many resonances are combinations of two or more normal plate modes (i.e., there is approximate degeneracy).

B. Sound spectra

The plates were given hard and soft blows from a soft mallet at several selected points, and the resulting model

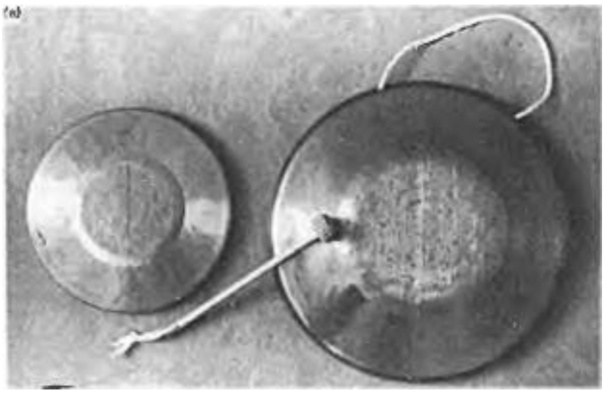
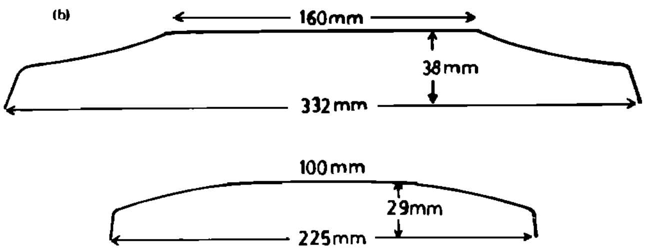

FIG. 2. (a) Photograph of Chinese opera gongs used in this study. (b) Dimensions of the gongs shown in (a).

frequencies were determined by means of a Hewlett–Packard 3582A spectrum analyzer. The frequencies at large and small amplitude of each principal mode of vibration were compared in order to determine the character of nonlinear behavior.

C. Results for Chinese opera gongs

Six different gongs of the type used in the "military" section of Chinese opera orchestras were studied. Although these gongs were fabricated by different gong makers in China, their behavior was fairly similar. The three larger gongs (300 to 332 mm in diameter) glide downward in pitch as much as three semitones after striking; the three smaller gongs (213 to 225 mm in diameter) glide upward in pitch as much as two semitones.

Two of the gongs are shown in Fig. 2 along with their dimensions. The central part of each large gong is essentially a flat circular plate with an average thickness of about 0.8 mm. The nearly conical shoulders have an average thickness about twice this value.

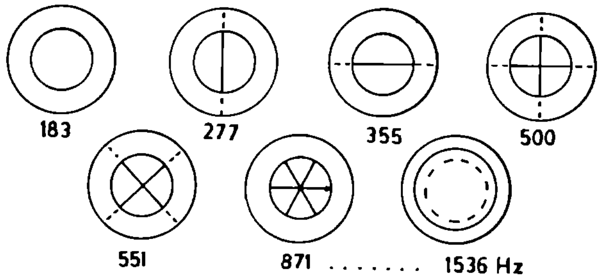

FIG. 3. Lowest modes of vibration of a Chinese opera gong.

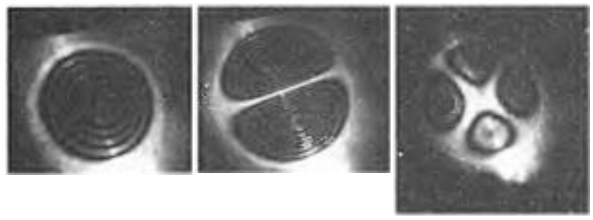

FIG. 4. Time-average hologram interferograms of the lowest three vibrational modes of a Chinese opera gong.[6]

The central part of each small gong, on the other hand, approximates a shallow spherical shell with an average thickness of about 0.7 mm. The shoulders are also roughly conical (with an average thickness of about 1.1 mm) but with more curvature than those of the large gongs.

The variation of thickness in some of the gongs was rather substantial; others were more nearly uniform. Thus the role of the thickness variation in determining the nonlinearity appears to be less important than suggested earlier.[6]

1. Modal frequencies

The modes of vibration of one of the larger gongs are shown in Fig. 3. The principal resonances are modes in which the bulk of the motion takes place in the central flat portion of the gong; the shoulders vibrate only a small amount in these modes. Note the large difference in frequency between the second and third modes, both of which have a single nodal diameter. This frequency difference is due to azimuthal variation in thickness. Time-average hologram interferograms of the first three modes in a small gong are shown in Fig. 4. Vibrations of small amplitude are mainly confined to the central part of the gong.[6]

Frequencies of the lowest mode of vibration excited by hard and soft blows are compared in Table I. [Recall that a frequency ratio of 1.06 (or 0.94) normally produces a pitch change of one semitone.] Gongs 1, 4, and 5 from Wuhan, China are examples of gongs widely used in the "military" sections of Chinese opera orchestras.

2. Effect of a static force

In order to investigate the effect of internal stress in the gongs it is necessary to vary this stress. We attempted to do this by changing the air pressure on one side of the gong while observing the change in modal frequencies. Fortunately, this was not too difficult to do, since the rim has negligible motion in the principal modes of vibration. Thus the gongs

TABLE I. Frequencies of the fundamental mode of vibration of Chinese opera gongs excited by hard and soft blows.

Gong	f (soft)	f (hard)	Ratio
1	180 Hz	212 Hz	1.18
2	176	212	1.20
3	376	404	1.07
4	472	430	0.83
5	472	390	0.91
6	700	640	0.91

were placed on a greased rubber gasket on a pump plate, and a vacuum pump was used to reduce the air pressure on the underside of the gong. The pressure difference p was measured with a mercury manometer. Furthermore, by spacing four to eight clamps (depending on the size of the gong) around the rim, it was possible to apply an overpressure of up to about 15 kPa without undue air leakage. With no pressure differential, the model frequencies of the clamped gongs were never more than 1% or 2% different from those measured with no clamps.

Results for several modes in a large gong are shown in Fig. 5. In this gong, both upward and downward forces were found to raise the modal frequencies. This is quite different from the behavior of the small gong shown in Fig. 6. In the small gong the frequencies rise under an upward force (positive pressure), but fall under a downward force (negative pressure) until a certain minimum frequency is reached.

In addition to providing an upward force on the central part of the gong a positive pressure also applies a radial tension due to the outward force on the shoulders. By the same token a negative pressure applies both a downward force and a radial compression due to inward force on the shoulders. It is difficult to determine the relative magnitudes of these forces in a gong and their individual effects on modal frequencies, however.

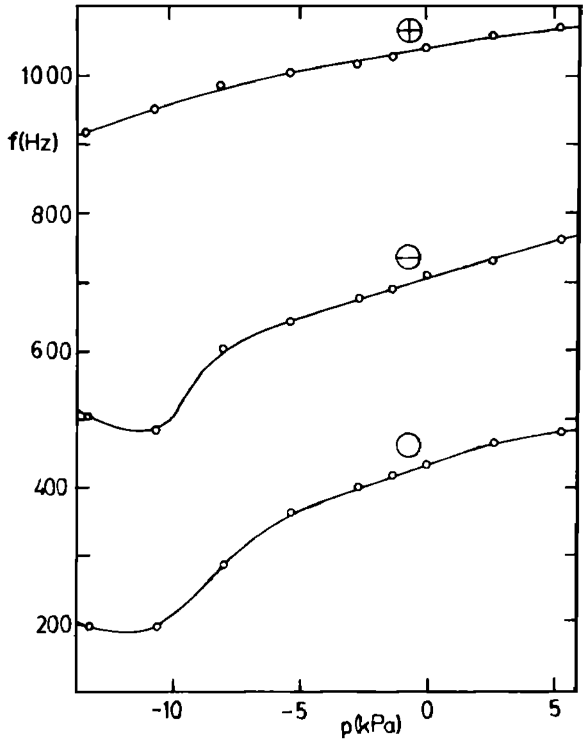

FIG. 6. Frequencies of three vibrational modes of a small Chinese opera gong under a static force.

D. Flat plate under compression and tension

In order to try to distinguish between the effects of static force and radial tension (or compression), the modes of a flat circular plate under tension and compression were studied. The apparatus used consisted of a flat circular plate 18 cm in diameter clamped near its circumference by 18 steel blocks; these blocks could be moved radially outward by means of bolts anchored to the heavy circular rim in order to apply radial tension or compression to the plate. Plates of two different thicknesses were studied.

The results obtained with a 0.5-mm-thick steel plate are shown in Fig. 7. The frequencies increase under tension and decrease initially under compression, in agreement with the

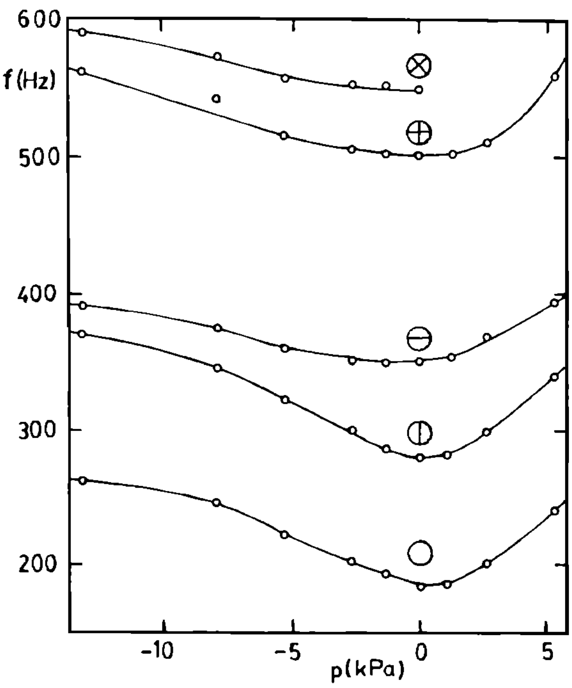

FIG. 5. Frequencies of five vibrational modes of a large Chinese opera gong under a static force. p is the air pressure inside the gong above and below atmospheric pressure.

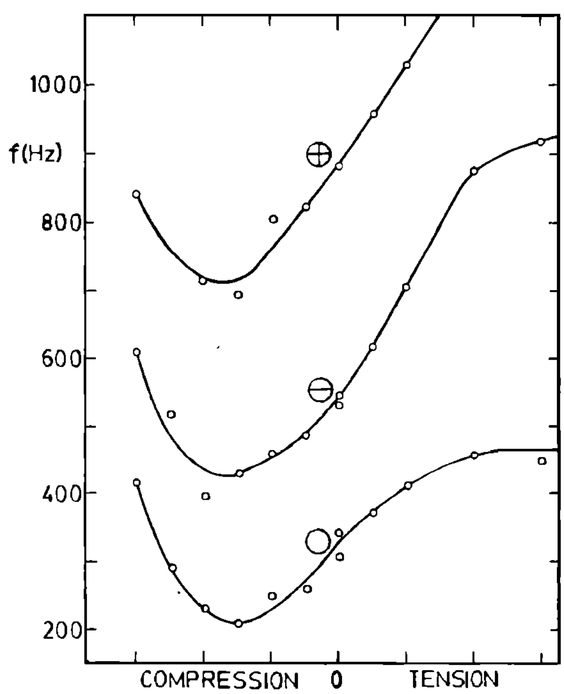

FIG. 7. Frequencies of three vibrational modes of a flat steel plate with radial compression or tension.

theoretical curves in Fig. 1. However, as the compression increases, the plate takes on curvature and the frequencies increase, as in the case of a shallow spherical shell. Unfortunately, we were not able to measure the actual tension and compression.

The nonlinear behavior of the plate was observed by striking it with blows of varying force. A slight softening (frequency change up to 5%) could be observed when the plate was under compression, and a smaller hardening effect (frequency change up to 2%) when under tension.

E. Gold pan

It was considered desirable to compare the behavior of the geometrically complex Chinese gongs with a similar plate of uniform thickness having a fairly simple geometry. A gold-washing pan was found to fulfill these requirements.

The gold pan we studied has a flat steel bottom 245 mm in diameter and 0.56 mm thick attached to nearly conical sides 105 mm long and 0.41 mm thick terminated in a rim with inner and outer diameters of 390 and 400 mm. About two-thirds of the way up the conical sides is a small groove, which does not appear to have any great effect on the vibrational behavior of the pan.

The principal vibrational modes of the gold pan were found to be of two types: (1) the "bell modes," which consist of flexural waves propagating around the conical portion of the pan; (2) the "dish modes" in which the bottom plate has motion similar to that of a vibrating flat circular plate. The modes of lowest frequency are shown in Fig. 8.

Striking the gold pan with a soft mallet however, produced a surprising result. When struck a sharp blow, the initial pitch of the gold pan appeared to be slightly below its small-amplitude value rather than above, as in the cases of a flat circular plate and the large Chinese gong (whose central portion is also flat). Study of the sound spectra revealed that the fundamental dish mode (which dominates the sound) was indeed about 5% lower in frequency when excited by a hard blow. It was further observed that pressing outward on the bottom (with one's fingers near the edge) raised the frequency by a considerable amount, whereas pressing inward lowered the frequency. In order to study this effect in more detail, the modes of vibration were studied with increased and decreased air pressure, as already described. Again the modal frequencies were virtually unaffected by clamping the inert rim against the gasket and pump plate.

Figure 9 shows the frequencies of the main dish modes under static pressure. Under reduced air pressure (inward force), the frequency of all the modes falls to a minimum and

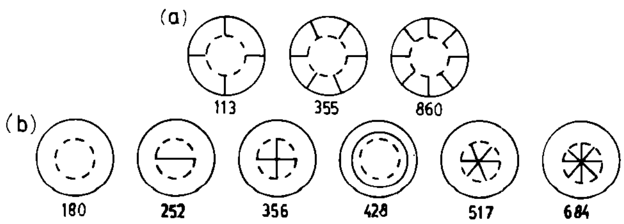

FIG. 8. Frequencies of five vibrational modes of a gold pan: (a) bell modes; (b) dish modes. The dashed circles outline the flat bottom.

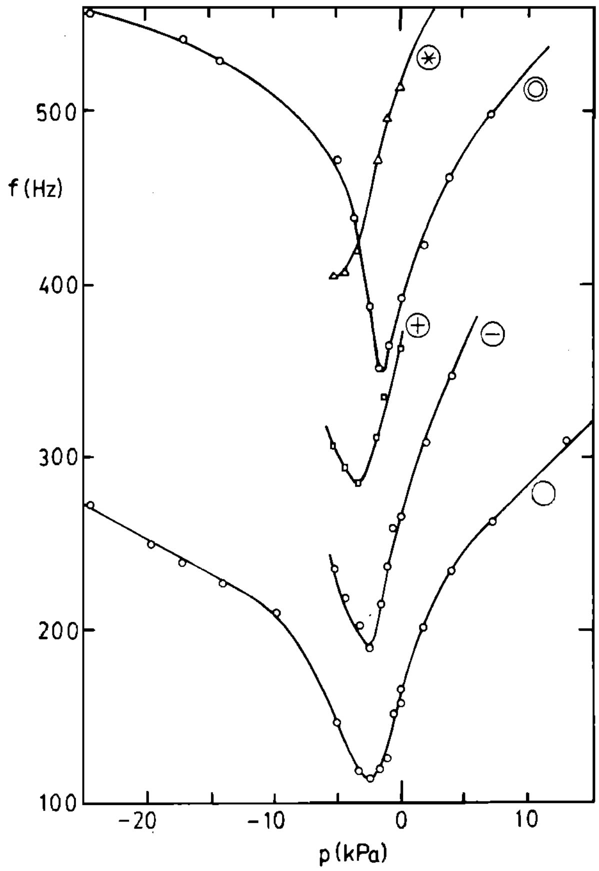

FIG. 9. Vibrational modes of a gold pan under static pressure. p is the air pressure inside the gold pan above and below atmospheric as in Figs. 5 and 6.

then rises. Outward force causes a frequency rise. Thus the circular bottom behaves as though it were under an outward tension, as will be discussed in a later section.

IV. DISCUSSION OF RESULTS

A. The gold pan

Reducing the air pressure inside the gold pan does two things: it causes the bottom to curve inward; and it pulls the conical shoulders inward. Pulling the shoulders inward apparently relaxes the outward tension on the bottom, and may even cause a radial compression. Thus the modal frequencies pass through minima as the air pressure inside the pan is reduced. At $p = -9$ kPa the bottom has curved inward about 1 mm, and this increases to about 4 mm at -25 kPa, the largest pressure difference applied.

Raising the air pressure inside the gold pan causes an outward curvature in the normally flat bottom, and it also increases the radial tension due to increasing force on the conical portion. An outward curvature of about 1.4 mm is observed at $p = 13$ kPa, which is slightly less than the inward curvature at the corresponding negative pressure, probably because of the outward tension.

The modal vibration frequencies of a flat steel plate having a clamped edge were calculated. The frequencies for a plate with the same thickness and diameter as the gold pan bottom are shown in Table II along with the frequencies observed for the dish modes of the gold pan which most

TABLE II. Comparison of the vibrational frequencies of a gold pan to those calculated for a flat circular plate with a clamped edge.

Mode	f (calculated)	$f(p=0)$	f (min)
(01)	91 Hz	180 Hz	116 Hz
(11)	189	252	190
(21)	310	356	284
(02)	354	428	358

nearly resemble them. Also shown are the minimum frequencies of these modes from Fig. 9. Note that the observed frequencies at atmospheric pressure are substantially greater than the calculated frequencies for the clamped plate, but the minimum frequencies show much closer agreement. It appears that the minimum frequencies, obtained by reducing the air pressure inside the gold pan, are nearly those of the flat circular bottom relieved of the radial tension it normally experiences (probably as a result of the manufacturing stamping process).

The behavior of the large-amplitude vibrations in the gold pan, particularly the slight nonlinearity of a softening type, can be partly understood by noting that in the fundamental mode the conical sides move in and out (in opposite phase to the flat bottom). If the bottom is under a radial tension, inward motion of the conical sides will lower the frequency in the same way as did reducing the air pressure under the pan (see Fig. 9), whereas outward motion will raise the frequency as did increasing the air pressure under the pan. The change of frequency with amplitude has the same sign as the second derivative of the curve of $(\text{frequency})^2$ versus pressure (see Appendix). Because the slope of the lowest curve in Fig. 9 is greater to the left of $p = 0$, the net result is a slight frequency shift downward at large amplitude (a nonlinearity of the softening type). If the bottom were not under a radial tension, an upward shift (nonlinearity of the hardening type) would have been expected, as in the larger Chinese gong shown in Fig. 5.

B. The Chinese opera gongs

The larger gong has a flat central portion which appears to be under little or no radial tension, since the minimum frequency in Fig. 5 occurs at atmospheric pressure (compare with the gold pan shown in Fig. 9, which also has a flat central portion). Thus the nonlinearity caused by striking is of the hardening type as in a flat plate.

The amplitude of vibration was measured by means of a pin adjusted in height so that it would just touch the plate during its greatest deflection. Amplitudes thus measured ranged from 0.6 to 0.9 mm in the large gongs and 0.6 to 0.8 mm in the small gongs, depending upon the manner of striking. These amplitudes are roughly 0.8 to 1.0 times the thickness of the central portion of the gong. From Fig. 1 of Ref. 11, the frequency shift in a circular plate with a clamped edge would be 12% to 17% for these amplitude/thickness ratios, and the corresponding shifts in a plate with a simply supported edge are 30% to 37%. Frequency increases of 18% and 20% were observed in the two large gongs (see Table I), which is considered satisfactory agreement, given the variation in thickness, uncertainty in internal stress, and the fact that the edge is neither clamped nor simply supported but rather something in between.

The smaller gong has a curved central portion which may be under an initial radial compression. From Fig. 4 of Ref. 5, in a shallow spherical shell with apex height H equal to the thickness, a vibration amplitude equal to the thickness will lower the frequency by about 5%. The observed decrease in frequency (Table I) ranges from 7% to 9% in the small gongs. Considering that the thickness varies considerably, and the central part of the gong only roughly approximates a spherical shell, the agreement between theory and experiment is again quite satisfactory.

As the air pressure inside the gong decreases, the curvature decreases, until it becomes nearly flat and eventually curves inward. This transition occurred at about $p = 9$ kPa in the gong shown in Fig. 6. At a pressure slightly below that, the frequencies of the lowest two modes appear to have reached minima in frequency.

V. CONCLUSIONS

The nonlinear behavior of plate vibrations appears to depend upon the curvature of the plate and the internal stress. A flat plate under no radial stress shows a nonlinearity of the hardening type at large amplitude (the frequency increases with amplitude). Most curved plates show nonlinearity of the softening type at large amplitude (the frequency decreases with increasing amplitude). Applying a radial tension to a flat plate increases the hardening behavior, whereas a radial compression can change the nonlinearity from a hardening type to a softening type.

These facts are sufficient to explain the nonlinear behavior of certain Oriental gongs, such as the ones used in Chinese opera orchestras. They were, no doubt, discovered empirically by generations of gong makers who learned to make gongs with either rising or falling pitch as desired.

ACKNOWLEDGMENTS

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APPENDIX

It is helpful to consider, in a semiquantitative way, the various physical situations that may occur in the nearly flat central section of a gong, and this is most simply done if this section is considered as a circular plate clamped around its periphery.

In Fig. A1 we show the forms expected for the curves relating central displacement y to deforming pressure difference p, as discussed in the text. Curve (a) shows the behavior of a simple, initially unstressed plate. The behavior is linear near the origin but, for large pressures of either sign, the elastic compliance of the plate, which is essentially the gradient of the curve, decreases because of well understood nonlinear effects, primarily extensional tension.

If the plate is under initial radial tension, then its behavior is represented by curve (b). This has essentially the same form as (a) but a lower compliance throughout. If the plate is subject to a small radial compression, then its behavior is of the form shown by curve (c), with a high compliance for small pressures but normal behavior for larger pressures.

For a larger radial compressive stress, the plate buckles to a dished shape and its behavior is represented by curve (d), which is symmetrical about the origin. The zero-stress configuration is represented by point A and, if the pressure is reduced below that corresponding to point B, the curvature of the dish reverses suddenly to point C. Similar hysteresis occurs in the opposite direction.

Finally, if the plate is not ideally thin, shear stresses may be incorporated into it during fabrication so that it maintains a dished configuration even in the absence of radial compressive stresses. The behavior of such a plate or shell is exemplified by curve (e), which has no equilibrium dished configuration of opposite curvature no matter what the pressure. Curves between (d) and (e) are of course possible.

Since the deformation of the plate under uniform pressure p is quite similar to the time-varying deformation produced during vibration in the fundamental mode, the frequency of this mode, for infinitesimal vibration amplitude, is to an adequate approximation inversely proportional to the square root of the slope of the characteristic curve at the point corresponding to the applied pressure

$$\omega_0(p) \propto \left(\frac{dy}{dp}\right)^{-1/2}. \tag{A1}$$

Cases (a)–(c) obviously produce $\omega_0(p)$ curves that are symmetrical about the origin, with ω_0 increasing as p increases in either direction, this change being most pronounced for case (c). In case (d), $\omega_0(p)$ decreases with decreasing p and approaches zero at point B, returns to a finite value as the dish inverts to C, and then increases as p is further decreased. In case (e) the frequency falls steadily to reach a minimum at the inflection point D and then increases again.

For a finite vibration amplitude Δy it is an adequate approximation to write for the frequency $\omega_\Delta(p)$

$$\omega_\Delta(p) \propto \left(\frac{\Delta y}{\Delta p}\right)^{-1/2}, \tag{A2}$$

where $\Delta y/\Delta p$ is simply the slope of a chord joining two points on the curve centered about p and a distance Δy apart.

In our experiments we have been concerned with the large amplitude situation only for $p = 0$. From Fig. A1 it is clear that, for cases (a)–(c), $\omega_\Delta(0)$ increases with increasing amplitude Δ. For cases (d) and (e), on the contrary, it seems clear that $\omega_\Delta(0)$ decreases with increasing Δ. A Taylor expansion of (A2) readily shows us that this latter conclusion is justified provided $d^3p/dy^3 < 0$ at the point A or E under consideration. Comparison with (A1) shows further that the sign

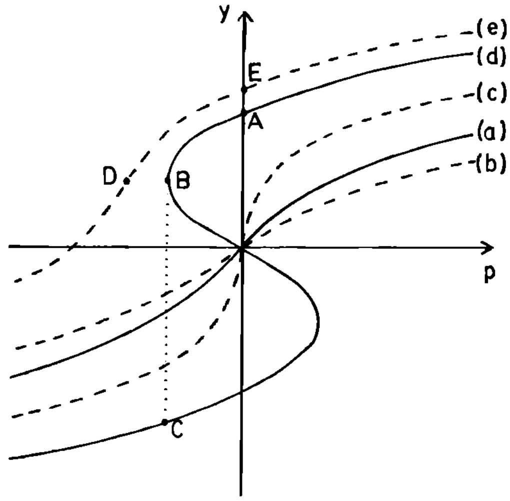

FIG. A1. Characteristic deformation curves under pressure p for a circular plate with fixed periphery: (a) unstressed; (b) with radial tension; (c) with radial compression; (d) with sufficient radial compression for dishing; (e) with inbuilt dishing stresses.

of $d\omega_\Delta/d\Delta$ is the same as that of $d^2\omega_0^2/dp^2$, which can be found from an appropriate replot of curves like those in Figs. 5, 6, or 9.

This brief analysis shows that our experimental results can be generally explained in this simple way, with the large gong corresponding to one of the cases (a)–(c) and the small gong to either (d) or (e), most probably the latter.

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