Nonlinear generation of missing modes on a vibrating string

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The nonlinear transfer of energy among modes of different frequencies on a vibrating string is investigated both theoretically and experimentally. The nonlinearity is associated with the well-known variation of string tension caused by the vibration modes, but it is essential that at least one of the end supports has finite mechanical admittance if there is to be any mode coupling. If the nonrigid bridge support has zero admittance in a direction parallel to the string, the coupling is of third order in the mode amplitudes. For a more realistic model in which the string changes direction as it passes over a bridge of finite admittance there are additional coupling terms of second order. The first mechanism gives driving terms of frequency $2\omega_n \pm \omega_m$ where ω_n and ω_m are, respectively, the angular frequencies of the nth and mth modes present on the string, while the second mechanism gives driving terms of angular frequencies $2\omega_n$ and $2\omega_m$. Analysis shows that modes absent from the initial excitation of the string can be driven to appreciable amplitude by these mechanisms, reaching their maximum amplitude after a time typically of order 0.1 s. Modes that are in nearly harmonic frequency relationship behave simply but coupling of modes that are appreciably inharmonic may give rise to rapid amplitude fluctuations. A simple experiment with a wire deflected by a bridge of elastic cord and plucked so as to eliminate a particular mode from the initial excitation provided general semiquantitative confirmation of the theoretical predictions.

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INTRODUCTION

It is well known from the standard analysis of the motion of stretched strings[1] that they can be excited in such a way that particular modes have zero amplitude. Thus, for example, for the case of an ideal string stretched between rigid supports, in which situation all the modes are harmonics of the fundamental, the nth harmonic and all its multiples mn ($m = 1,2,3,\ldots$) are absent if the string is excited by plucking or striking it at a point $1/n$ of its length from one end.

These conclusions are modified only in detail if the theory is extended to include the stiffness of a real string or the incomplete rigidity of real end supports, both of which make the modes of a real string slightly inharmonic.[2,3] It is always possible, according to the standard linear theory, to eliminate a particular mode from the motion by applying the excitation at one of the nodes of that mode or, more generally, in such a distribution that the excitation function is orthogonal to the mode shape function.

It comes as something of a surprise, therefore, to find that in practical cases, for example in musical instruments with plucked or hammered strings, these modes are not actually absent from the motion. Rather, they typically begin with near-zero amplitude, rise to a peak after a time of the order of 0.1 s, and then decay.

It is the purpose of the present paper to investigate this phenomenon, in an appropriately idealized situation, both theoretically and experimentally. We shall find that the explanation and its quantitative treatment are both relatively straightforward and give insight into a variety of more complex vibration processes.

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I. THEORY FOR AN IDEALIZED BRIDGE

It is clear from quite general considerations that the phenomena described can result only from the operation of nonlinear mechanisms. In a linear system the normal modes are uncoupled and, in the presence of viscous damping forces, each mode will simply decay with its own characteristic lifetime.

The stretched string, like most other physical systems, is linear to only a first approximation; the major cause of nonlinearity is the fact that any small transverse displacement of the string makes a second-order change in its length, and therefore in its tension. This has long been recognized, and Carrier[4] has given a detailed solution for the steady motion of a string with rigid supports. A more recent discussion from a different viewpoint has been given by Murthy and Ramakrishna.[5] These treatments, however, do not give any immediate help in solving the present problem. Instead it turns out to be more appropriate to start again from first principles and to construct the solution to the rather more general nonlinear problem in which the supports are not completely rigid, maintaining only the amount of rigor that is essential.

Consider the string shown in Fig. 1(a). It is stretched with tension T between two supports, one at $x = 0$ which is rigid and one at $x = L$ which is rigid in the x direction but which has a mechanical admittance (velocity/force) equal to $Y_B(\omega)$ in the y direction at angular frequency ω. Such an arrangement is somewhat analogous to that of a string on a piano, harpsichord, or guitar, the compliant support being the bridge that is attached to the soundboard. There are, however, important differences between this idealized situation and a more realistic model of a musical instrument

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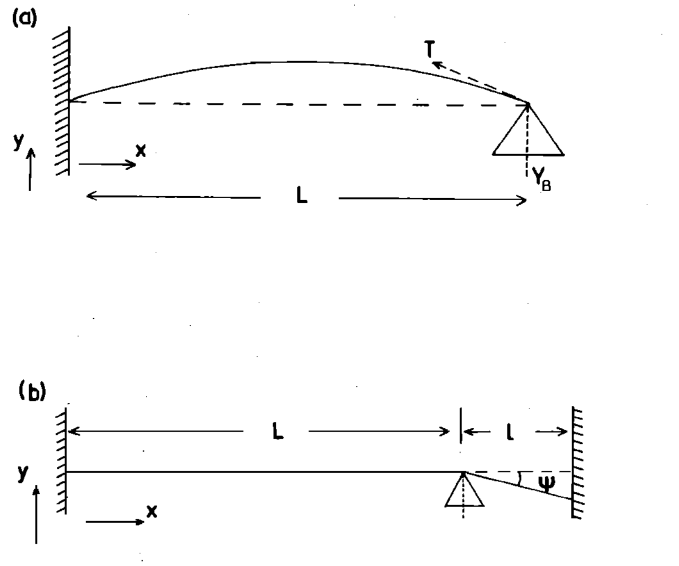

FIG. 1. The system to be analyzed. A flexible string under tension T is attached to a rigid support at $x = 0$, and its other end is attached to a bridge with lateral mechanical admittance Y_B. (a) shows a highly idealized system while (b) is a closer approximation to reality.

bridge such as shown in Fig. 1(b). We will return to this point later.

If we assume, as is usually true in practice, that the tension is low enough such that the velocity of transverse waves on the string is very much less than that of longitudinal waves,[5,6] then the tension T can be taken as uniform along the string, and the displacement y, assumed to lie in a single plane as is appropriate for our experiment to be described later, obeys the equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} - D \frac{\partial y}{\partial t}, \tag{1}$$

where ρ is the mass of the string per unit length, and the coefficient D is a measure of the viscous losses to the surrounding air and internal to the string itself.

The elementary solution for this problem is well known[1] when both supports are rigid, so that $Y_B(\omega) = 0$. The eigenfunctions have the form

$$y_n(x,t) = a_n \sin\left(\frac{n\pi x}{L}\right) \sin(\omega_n t + \phi_n) \exp\left(\frac{-t}{\tau_n}\right), \tag{2}$$

where

$$\omega_n = \left[\left(\frac{n\pi c}{L}\right)^2 - \left(\frac{1}{\tau_n}\right)^2\right]^{1/2} \approx \frac{n\pi c}{L},$$
$$\tau_n = 2\rho/D(\omega_n), \quad c = (T/\rho)^{1/2}. \tag{3}$$

Finite stiffness of the string raises all angular frequencies ω_n and introduces further inharmonicity.[2] But since this effect is both small and nearly independent of amplitude, in the interest of algebraic simplicity we ignore it in our discussion.

The fundamental nonlinearity arises because of the second-order change in the length of the string with vibration amplitude. This length increase is given by

$$\Delta = \int_0^L \left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} dx - L \tag{4}$$

so that the tension becomes

$$T = T_0 + ES\Delta/L, \tag{5}$$

where E is the Young's modulus of the string material and S is the string's cross-sectional area. Inserting a sum $y = \Sigma y_n$ of the form (2) into (4) and neglecting terms of fourth or higher order in a_n/L gives

$$T \approx T_0 + \frac{\pi^2 ES}{8L^2} \sum_n n^2 a_n^2 \times [1 - \cos 2(\omega_n t + \phi_n)] \exp(-2t/\tau_n). \tag{6}$$

The first part of the sum in this expression gives a quasistatic increase in the tension, varying as $\Sigma\, n^2 a_n^2 \exp(-2t/\tau_n)$. This causes, from (3), a proportional increase in the frequencies of all the modes, the increase dying away with time, and is responsible for the characteristic twang of vigorously plucked strings, particularly if they are of metal rather than gut or nylon, so that the Young's modulus E is high.

The remaining terms in the summation contribute oscillatory components of the tension, the frequency $2\omega_n$ being associated with mode n. Suppose the string also carries mode m, then, since the transverse driving force associated with tension is $T\,\partial^2 y/\partial x^2$, the driving force exerted by mode n on mode m has a time variation at frequency $2\omega_n \pm \omega_m$. Following this through, we find that this can influence mode m only if $n = m$. Even the driving force at frequency $2\omega_n + \omega_m$ cannot, however, influence the mode near that frequency because the spatial distribution along the string is orthogonal to it. Thus the various modes can act back only upon themselves by this mechanism.

We conclude that the mode conversion effects that we wish to understand should be absent from a string supported between two rigid bridges. We therefore turn to the case in which the admittance $Y_B(\omega)$ of the support at $x = L$ is finite.

In the linear approximation, the termination condition at the nonrigid support is just

$$\left.\frac{\partial y}{\partial t}\right|_{x=L} = -Y_B T_0 \left.\frac{\partial y}{\partial x}\right|_{x=L}. \tag{7}$$

It would be possible to apply this condition without restriction to the lossy string described by Eq. (1), but it helps the clarity of the argument and is also adequate for our practical purposes to make some simplifying assumptions. The first of these is that the bridge admittance $Y_B(\omega)$ is always small compared with the characteristic admittance $1/\rho c$ of the string, so that the bridge is nearly a node for the string motion. The second is that the energy losses in the system can be partitioned between those at the bridge, giving decay time constants τ_n', and those associated with viscous or internal losses along the string, giving decay time constants τ_n'', so that

$$\frac{1}{\tau_n} = \frac{1}{\tau_n'} + \frac{1}{\tau_n''}, \tag{8}$$

where it is now τ_n'' rather than τ_n that is related to $D(\omega_n)$ by (3). The third is to assume that the bridge itself behaves as a simple linear system so that $Y_B(\omega)$ can be written as an amplitude-independent complex function of ω when the time-variation of the eigenfunctions (2) is written as $\exp(j\omega t)$.

With these assumptions we can adapt the linear analysis given, for example, by Morse[3] to derive the results

$$\omega_n \approx (n\pi c/L) - (T_0/L)\mathrm{Im}(Y_B), \qquad (9)$$

$$\tau_n' \approx (T_0/L)\mathrm{Re}(Y_B), \qquad (10)$$

where Re (Y_B) and Im (Y_B) are, respectively, the real and imaginary parts of the bridge admittance Y_B.

The eigenfunctions, for the linear approximation, still have the form (2), with (9) and (10) inserted and with L replaced by $L+\delta_n$ where

$$\delta_n \approx (T_0/\omega_n)\mathrm{Im}(Y_B). \qquad (11)$$

The string thus behaves, for each mode, very much like a normal lossy string supported rigidly at $x=0$ and $x=L+\delta_n$.

For the more general nonlinear problem to which we now turn, the force on the compliant bridge is, to sufficient accuracy from (2) and (6),

$$F = -T\frac{\partial y}{\partial x}\bigg|_{x=L} \approx \Big(T_0 + \sum_n T_n\Big)\sum_m A_m \sin\theta_m - \frac{1}{2}\sum_n\sum_m T_n A_m \sin(\theta_m \pm 2\theta_n), \qquad (12)$$

where

$$T_n \approx (\pi^2 ES/8L^2)n^2 a_n^2 \exp(-2t/\tau_n), \qquad (13)$$

$$A_n \approx (-1)^{n+1}(\pi/L)na_n \exp(-t/\tau_n), \qquad (14)$$

$$\theta_n \approx \omega_n t + \phi_n, \qquad (15)$$

the approximations arising from neglecting δ_n in comparison with L in appropriate places.

The force F acts upon the compliant bridge and its attached string, which are mechanically in series since they have the same displacement velocity. To the approximation to which we are working, the problem is thus equivalent to that of a rigidly supported string of length $L+\delta$, acted upon by the force F at the point $x=L$, which is of distance δ from one end. The resistive part Re (Y_B) of the bridge admittance may also be localized at this point.

The equation of motion for the string is now formally, comparing with (1),

$$\rho\frac{\partial^2 y}{\partial t^2} = T_0\frac{\partial^2 y}{\partial x^2} - D\frac{\partial y}{\partial t} - R'\frac{\partial y}{\partial t}\delta(x-L) + F(t)\delta(x-L), \qquad (16)$$

where $y=0$ at $x=0$ and $x=L+\delta_p$ for the pth mode, the form of which is given by (2) with L replaced by $L+\delta_p$ and with δ_p given by (11). The form of R', which is clearly related to τ_p by (10), need not concern us for the moment.

Multiplying (16) by the spatial part of y_p, namely $\sin[p\pi x/(L+\delta)]$, and integrating over $x=0$ to $L+\delta$ gives

$$\rho\frac{d^2 z}{dt^2} = -T_0\Big(\frac{p\pi}{L+\delta}\Big)^2 z - R\frac{dz}{dt} + \frac{2}{L}F(t)\sin\Big(\frac{p\pi L}{L+\delta}\Big), \qquad (17)$$

where R is related to D and R', and the pth normal mode is now represented by

$$z = a_p \sin(\omega_p t + \phi_p) \qquad (18)$$

when $F=0$. This serves, if we wish, to relate R to τ_p. Using (3), assuming $\delta \ll L/p$, and then using (11), we can rewrite (17) as

$$\frac{d^2 z}{dt^2} + \omega_p^2 z = -\frac{2}{\tau_p}\frac{dz}{dt} - (-1)^p\frac{2p\pi T_0}{\rho\omega_p L^2}\mathrm{Im}(Y_B)F(t). \qquad (19)$$

There is an equation of this form for each of the modes p. In cases of practical interest, the damping is small so that $\tau_p \ll 1/\omega_p$ and the resonances are so narrow that only those terms in $F(t)$ with frequencies very close to ω_p need to be considered. We note, incidentally, that the forcing term involving $F(t)$ is proportional to the bridge admittance Y_B, and so vanishes for a string supported on rigid bridges, in agreement with our earlier discussion.

II. FIRST-ORDER SOLUTION

The differential equation (19) for the pth mode has the general form

$$\frac{d^2 z}{dt^2} + \omega_p^2 z = g(t), \qquad (20)$$

where the function g contains both damping and forcing terms and depends upon the amplitudes of all the other mode vibrations. This equation is in the standard form for treatment by the method of slowly varying parameters[7,8] in which each of the modes z is assumed to have the form

$$z = a_p(t)\sin[\omega_p t + \phi_p(t)], \qquad (21)$$

where a_p and ϕ_p are slowly varying functions of t. The further assumption that

$$\frac{dz}{dt} = a_p(t)\omega_p \cos[\omega_p t + \phi_p(t)] \qquad (22)$$

establishes a limitation on the allowed forms of a_p and ϕ_p. Finally, with (21) and (22) substituted in (20), the resulting equation is multiplied by $\cos\theta_p$ or $\sin\theta_p$, with θ_p given by (15), and averaged over one period $2\pi/\omega_p$, all terms being neglected except those that change slowly over this time. The resulting equations are

$$\langle \dot a_p\rangle = \frac{1}{2\pi\omega_p}\int_0^{2\pi} g(t)\cos\theta_p\, d\theta_p = \sum_n\sum_m \beta_{mnp} a_m a_n^2 \sin[(\omega_m \pm 2\omega_n \pm \omega_p)t + (\phi_m \pm 2\phi_n \pm \phi_p)] - a_p/\tau_p, \qquad (23)$$

$$\langle \dot\phi_p\rangle = -\frac{1}{2\pi\omega_p a_p}\int_0^{2\pi} g(t)\sin\theta_p\, d\theta_p = \sum_n\sum_m \pm\Big(\frac{\beta_{mnp} a_m a_n^2}{a_p}\Big)\cos[(\omega_m \pm 2\omega_n \pm \omega_p)t + (\phi_m \pm 2\phi_n \pm \phi_p)], \qquad (24)$$

where only the terms $m \pm 2n \pm p = 0$ are retained. In these expressions the $\pm$ before ω_p is independent of that before $2\omega_n$ but tied to the leading $\pm$ in (24). The signs of the ϕ_j follow those of their related ω_j. The mode coupling coefficients β_{mnp} follow directly from (12)–(14), and (19).

A solution to the general problem requires consideration of the pairs of coupled equations (23) and (24) for all modes of the string. For our present problem, however, a much simpler procedure suffices. Suppose the string is excited so that only two modes n and m have appreciable amplitude, and a third mode of interest, p, has zero initial amplitude. We further suppose that modes n and m are either not coupled, in the sense that $m \pm 2n \neq m$, $3n \neq m$, and similarly with m and n interchanged, or that any such coupling is small enough to be neglected. Then the forcing term F will be zero at mode frequencies ω_n and ω_m, except for some small self-interactions, and each of these modes will simply decay exponentially

$$a_n = a_n^0 \exp(-t/\tau_n), \tag{25}$$

$$a_m = a_m^0 \exp(-t/\tau_m). \tag{26}$$

If n and m are chosen so that one or more of the combinations $|m \pm 2n|$, $|n \pm 2m|$ (including $3n$ or $3m$) is equal to p, however, then the forcing term F for mode p will not vanish.

In the simple case in which ω_m and ω_n are harmonically related, the frequencies of these driving terms near ω_p will be the same, although not necessarily exactly equal to ω_p unless it too is part of the same harmonic relationship. Provided the discrepancy between ω_p and the driving frequency is not too large, the phase ϕ_p of mode p (which is initially undetermined) can adopt a value such that the actual oscillation frequency ω_p', given by (21) as

$$\omega_p' = \omega_p + \langle \dot{\phi}_p \rangle \tag{27}$$

is equal to the driving frequency. This is simply the phenomenon of off-resonance driving as expressed in this formalism. Once the initial phase ϕ_p has been determined in this way, $\langle \dot{\phi}_p \rangle$ remains nearly constant, only tracking the quasistatic tension change, and the value of $\langle \dot{a}_p \rangle$ can be determined. There is an inverse relationship between the magnitudes of these two quantities, given explicitly by (23) and (24), as is always the case for systems slightly away from resonance.

The general problem requires numerical methods at this stage unless the relevant frequencies ω_n and ω_m are harmonically related, for the general solution will involve beats and similar fluctuation phenomena. Suppose, however, that we can neglect all driving modes except m and n, driving mode p through the coefficient β_{mnp}, and let

$$\Delta\omega_p = \langle \dot{\phi}_p \rangle = \omega_p' - \omega_p. \tag{28}$$

Then from (24), in abbreviated notation,

$$\cos[\;] = \pm \Delta\omega_p a_p / \beta_{nmp} a_m a_n^2, \tag{29}$$

and this can be substituted back into (23) to give

$$\langle \dot{a}_p \rangle = \beta_{mnp} a_m a_n^2 \left[1 - \left(\frac{\Delta\omega_p a_p}{\beta_{nmp} a_m a_n^2} \right)^2 \right]^{1/2} - \frac{a_p}{\tau_p}. \tag{30}$$

It is clear from the general form of (30) that, if a_m and a_n are both initially proportional to some pluck amplitude A while a_p is zero, then a_p will rise at a rate proportional to A^3, go through a maximum as a_m and a_n decay, and then decay itself with a time constant tending towards τ_p. It is also clear that the excitation of mode p will be most efficient when all modes are nearly harmonically related, so that $\Delta\omega_p$ is small, and when the decay times τ are long.

Another conclusion that follows from the general form of (30) is that not all modes can be driven by this mechanism but only those for which $p = |2n \pm m|$. In particular if the modes are essentially harmonic and the string is plucked near its center so that no even modes are excited, then the mechanism cannot provide subsequent excitation of any of these even modes.

III. THEORY FOR A REALISTIC BRIDGE

The idealized bridge shown in Fig. 1(a) and considered in our theory above differs significantly from the more realistic bridge shown in Fig. 1(b). For the realistic bridge structure, it is an adequate approximation for our present purpose to assume that the bridge itself is quite compliant in a direction parallel to the string, with the necessary longitudinal rigidity being provided by the short angled length of string between it and the hitch-pin. We can then assume that the tension T of the string is the same throughout its whole length.

The analysis of the previous section still applies to this structure, with the admittance Y_B being interpreted as applying to the transverse behavior of the bridge and string tail together. However, as we shall see in a moment, there is also a further nonlinearity of second rather than third order which can dominate the behavior in certain cases.

Suppose that the bridge is in equilibrium under tension T_0 and that this tension is increased to the value T by the mechanism leading to Eq. (7). Then, because of the inclination ψ of the tail of the string, this increase leads to an additional downwards force F on the bridge structure of magnitude

$$F(t) = (T - T_0) \sin \psi. \tag{31}$$

If the string carries modes m and n, then, from (6), $F(t)$ has components of frequencies $2\omega_m$ and $2\omega_n$ with amplitudes proportional to a_m^2 and a_n^2, respectively. There is no frequency mixing below fourth-order terms.

The effect of this force can be treated in exactly the same way as before, and the governing equation is formally identical with (16) or (19). The solution is given by equations similar to (23) and (24) or, quite generally since there is only one driving term for each mode, by an equation like (30). The only difference is the replacement

$$\beta_{mnp} a_m a_n^2 \to \beta'_{np} a_n^2 \sin \psi, \tag{32}$$

where the detailed form of β'_{np} is given by the algebra, and the only slowly varying terms are those for which $p = 2n$.

In summary we note that this nonlinearity can drive only even modes, that it behaves as the square rather than the cube of the initial excitation amplitude, and that its coupling magnitude is proportional to $\sin \psi$, where ψ is the bending angle of the string as it passes over the bridge.

IV. EXPERIMENT

The experimental arrangement consisted of a tensioned nichrome wire, 0.3 mm in diameter, passing over two rigid bridges about 67 cm apart. The fundamental frequency was about 200 Hz. A horseshoe permanent magnet produced a strong magnetic field normal to the wire over a short part of its length, and the emf induced by motion gave a signal proportional to appropriately weighted amplitudes of the string

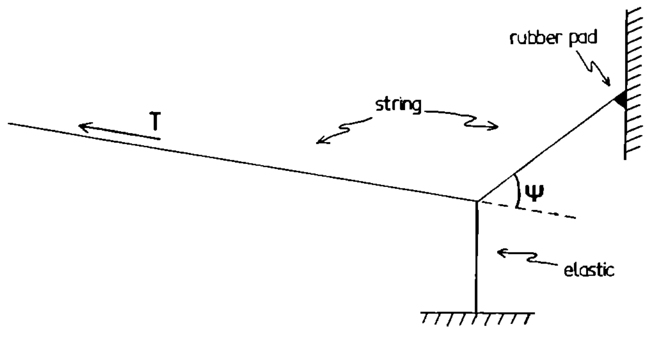

FIG. 2. The compliant bridge used in the experiment. Deflection angles are exaggerated for clarity.

vibration modes. The modes were isolated by feeding the signal through a digital filter (Bruel and Kjaer type 1623) with a bandwidth of 12% or 24%, thus giving an adequately short rise time, and were displayed on a storage oscilloscope. The relative response of the system to different modes was easily calculated and this calibration was checked and made absolute by plucking the string through a well-defined displacement.

The first check of theory for rigid bridges was performed by plucking the string to a displacement of about 3 mm at one-third of its length, giving large amplitudes to modes 1 and 2 and a carefully defined near-zero amplitude to mode 3. No subsequent increase in the amplitude of mode 3 was observed, in agreement with theoretical prediction. A similar confirmatory null result was found for mode 2 when the string was plucked at its center.

A compliant bridge was constructed by looping a thin, cotton wrapped elastic cord over the wire and securing it to a lower support as shown in Fig. 2. This arrangement gave a bridge that was not only nearly ideally compliant but also very substantially anisotropic, thus effectively decoupling the two polarizations of wire vibrations and eliminating one source of possible experimental difficulty.[5] A small pad of rubber damped the short end of the string and minimized undesired high-frequency vibrations without adding a significant resistive component to the bridge admittance. The string length L was about 55 cm, the tail length l about 12 cm, and the angle ψ about 3°.

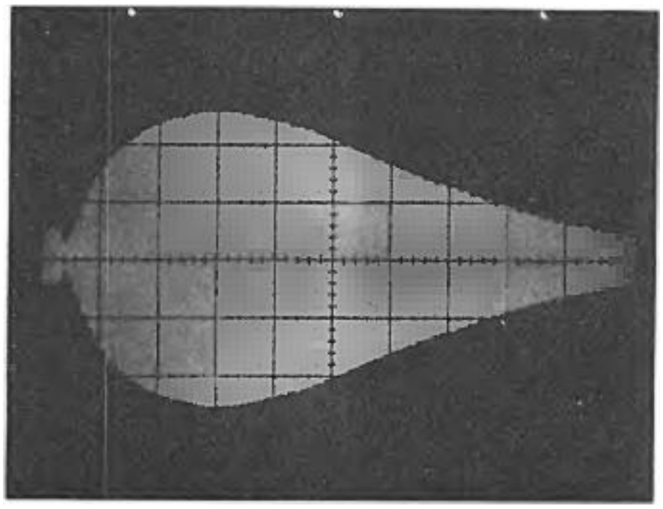

FIG. 3. Oscilloscope record of the growth and decay of the third vibration mode of the string when it is plucked at a node for that mode. The major graticule divisions are 0.05 s apart.

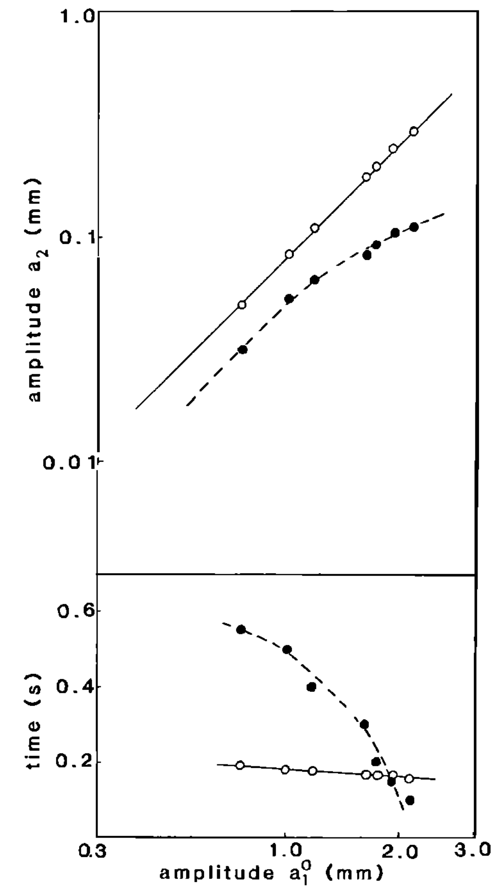

FIG. 4. Measured values (filled circles, broken curve) and calculated values (open circles, full curve) of the maximum amplitude a_2 achieved by the second mode, and of the time to reach this amplitude, as functions of the initial amplitude a_1^0 of the first mode.

With this arrangement and using a mechanical plucking device, several series of plucks were recorded on magnetic tape for later analysis. In one series the string was plucked close to its midpoint in a position found to give nearly zero excitation of the second mode, while in the other series a plucking point near one-third of the string length was used so as to minimize the initial amplitude of the third mode. In each case a range of pluck amplitudes up to about 3 mm was used.

When the records were replayed for analysis, the modes initially excited were found to decay more or less exponentially with time, while the unexcited modes grew from near zero to a maximum in a time of order 0.1 s and then decayed slowly to zero. A typical trace is shown in Fig. 3. To analyze these results, the peak amplitude reached by the missing mode and the time delay to this peak were both plotted as functions of the inital fundamental mode amplitude a_1^0, also derived from the recording. These measurements are shown in Fig. 4 for the second mode and in Fig. 5 for the third mode.

Comparison with theory must await the discussion of the next section, but we see immediately that the peak amplitude of the "missing" mode, though substantially less than the initial amplitude of the fundamental, is of quite signifi-

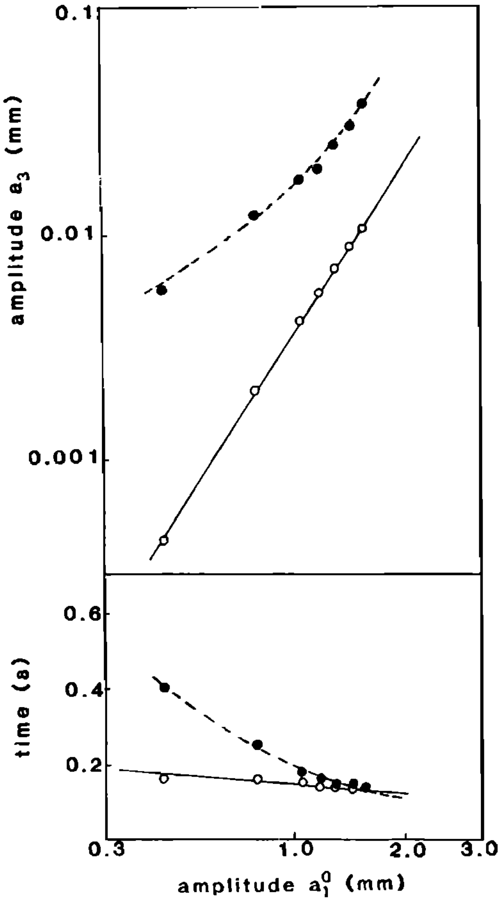

FIG. 5. Measured values (filled circles, broken curve) and calculated values (open circles, full curve) of the maximum amplitude a_3 achieved by the third mode, and of the time to reach this amplitude, as functions of the initial amplitude a_1^0 of the first mode.

cant magnitude. The amplitude of this peak does not behave in an entirely simple manner in its dependence upon the amplitude a_1^0 of the fundamental in either case, though clearly the maximum values of both a_2 and a_3 increase with increasing a_1^0. The time delay to the peak amplitude decreases significantly with increasing a_1^0 in both cases, the measured range of variation being as much as a factor of 3 over the amplitude range studied.

V. ANALYSIS

In the experimental arrangement, the bridge consists effectively of the elastic cord together with the oblique end of the string. The static compliance of this combined bridge was found by careful measurement of geometry, followed by a measurement of the amount by which the string deflection at the bridge changed when the string tension was lowered or raised. The behavior was linear within the accuracy of measurement, and the resulting numerical value was

$$\lim_{\omega \to 0} \omega^{-1} \operatorname{Im}(Y_B) \approx 0.6 \text{ mm N}^{-1}. \tag{33}$$

If the real part of Y_B is small, as we will see in a moment that it is, then this implies from (11) a δ value of about 2 cm which does not change very much with frequency. This was checked by noting the change in frequency when the compliant bridge was replaced by a rigid bridge without disturbing either geometry or string tension, and the agreement with calculation was found to be good. The frequencies of the first three modes are harmonically related to a good approximation.

Experiments showed that the insertion of a small rubber pad at the remote end of the string tail made little difference to observed decay times, while the decay time for the string with its compliant bridge is about half that for the same string supported between two rigid bridges. Since the end correction δ is not very sensitive to the exact value of τ', it is adequate to assume $\tau_n' = \tau_n'' = 2\tau_n$ for all n. Use of Eq. (1) together with measured τ values and compliances for the complete bridge structure allowed two values, respectively, of order 10^{-2} and 10^2 N m^{-1} s, for the effective resistance R in series with the bridge compliance. The smaller value is supported by the relatively large value of δ referred to above. An independent measurement of the mechanical admittance of a similar stretched piece of the same elastic cord using a Bruel and Kjaer type 8001 impedance head gave an effective series resistance of a few times 10^{-3} N m^{-1} s at 100 Hz, confirming the lower resistance value. This resistance was found to increase with increasing vibration amplitude, an effect which is shown also by the resistance deduced from the τ values, as we shall discuss again below.

To calculate approximately the frequency variation of the bridge admittance Y_B, an analog model of the form shown in Fig. 6 was used. The 4-pole network Z_{ij} represents the tail of the string, the large resistance R'' the effect of the rubber pad at the rigid bridge, and the combination R,C the behavior of the elastic cord. Now R'' is very large compared with the characteristic impedance ρc of the string, as is shown by the negligible effect of R'' on the modes in which we are interested. We can therefore set $R'' = \infty$ to adequate accuracy and replace Z_{ij} by its input impedance

$$Z_{11} = -j\rho c \cot(\omega l/c), \tag{34}$$

where l is the length of the string tail. The known low-frequency limit of $\operatorname{Im}(Y_B)$ given by (33) and the known value of τ in (10) to give $\operatorname{Re}(Y_B)$ then allow all the quantities in the model of Fig. 6 to be calculated.

We must now introduce a further significant nonlinearity which has so far been neglected. This relates to the damp-

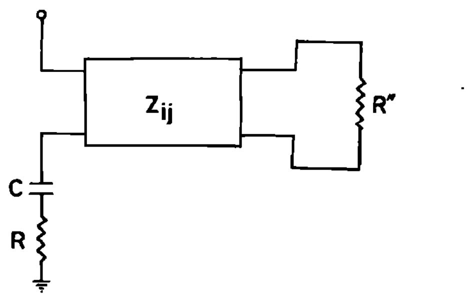

FIG. 6. Electrical analog circuit for the mechanical properties of the bridge used in our experimental arrangement. R and C represent the behavior of the elastic cord, Z_{ij} is the impedance of the tail of the string, and R'' the large resistance with which it terminates. The input admittance of this network is Y_B.

ing term $-a_p/\tau_p$ in (23) and (30) which accounts for the free decay of mode p. This linear approximation may indeed be adequate in some practical situations, but measurements of the free decay of modes excited singly in our experimental arrangement showed that the time constant τ_p depended significantly upon mode amplitude a_p, with τ_p decreasing considerably for large a_p. Such an effect is indeed expected for the viscous drag exerted on the string by the surrounding air, since the Reynolds number for our vibrating string lay typically in the range 10 to 100, which is just the range over which the drag for steady flow past a cylinder changes from linear to quadratic.[9] In our case, however, much of the damping is caused by the bridge structure, for which quite different considerations apply. Nevertheless, the form of equation suggested by the air-damping analogy, namely

$$\tau_p^{-1} = \alpha_p + \gamma_p a_p \tag{35}$$

was found to fit the data quite adequately, with a_p and γ_p as experimentally determined constants.

Clearly with several modes present on the string simultaneously, as in the experimental plucked situation, we would expect interaction terms in the damping. This was confirmed by examining the decay of a particular mode when the string was plucked rather than excited only in that mode. The experimental results are adequately approximated by the obvious generalization

$$\tau_p^{-1} \approx \alpha_p + \gamma_p a_p + \sum_{n \neq p} \gamma_{pn} a_n, \tag{36}$$

where only the one or two most important terms are included in the summation. Experimental values for the relevant α and γ parameters for our particular experimental arrangement are given in Table I.

For the purpose of our numerical solutions, the form (36) was simply inserted into (9) and (10), using the analog circuit of Fig. 6, to determine the ω_n and hence $\Delta\omega_p$ at each instant, and then into (30) or (32) for numerical integration to plot the behavior of the missing mode amplitude a_p. The averaged nature of the expression (36) does not justify any more detailed treatment.

In Fig. 7 we show the calculated behavior of the third mode amplitude, when the string is plucked so as to make its initial value zero, for the particular string and bridge configuration used in our experiments. The calculated behavior of the second mode when it is initially unexcited is very similar. Clearly the general predictions of the theory are qualitatively similar to the experimental behavior shown in Fig. 3.

For quantitative assessment of the theory, its predictions of maximum amplitude and time to reach that amplitude are shown by open circles plotted alongside the experimental data (filled circles) in Figs. 4 and 5. The absolute values of the predicted amplitudes are of the right order in each case, and these amplitudes increase with the amplitude of the exciting fundamental with something close to the observed slope. More specifically, the calculated slope for $\log a_2$ against $\log a_1^0$ is about 1.6 and for $\log a_3$ against $\log a_1^0$ is about 2.2. The general slope of the experimental curves is in each case rather less than this. The calculated times to reach maximum amplitude are similarly of the observed order of magnitude but are significantly too small at small pluck amplitudes and show much too little variation with amplitude.

TABLE I. Damping parameter α in $(\mathrm{s})^{-1}$ and parameter γ in $(\mathrm{s\ mm})^{-1}$ for the experimental system.

Mode 1	Mode 2	Mode 3
$\alpha_1 = 0.4$	$\alpha_2 = 0.7$	$\alpha_3 = 0.08$
$\gamma_1 = 0.6$	$\gamma_2 = 2.5$	$\gamma_3 = 9$
$\gamma_{21} = 2.5$		
$\gamma_{31} = 6$	$\gamma_{32} = 12$	

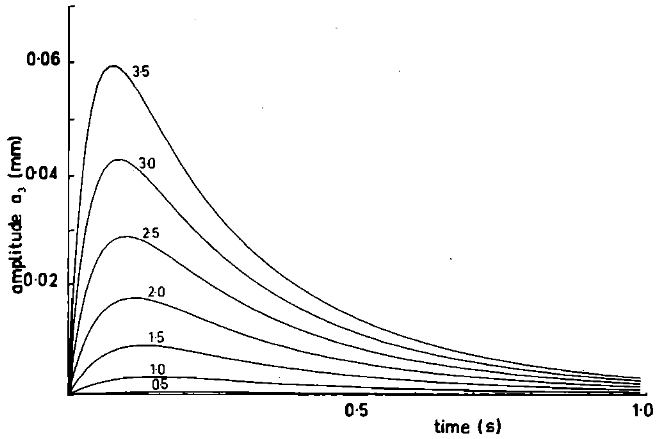

FIG. 7. Calculated growth and decay of the third mode of the string, plucked at a node for that mode, for first mode amplitudes ranging from 0.5 to 3.5 mm (shown as a parameter).

It is thus apparent that, while the theory can account for the observed phenomena in a general way, it is only in semi-quantitative agreement with experiment. The most likely explanation for the residual disagreements is the failure of our experimental setup to provide an adequate approximation to the idealized situation assumed for the theoretical analysis. This applies in particular to the behavior of the bridge admittance. It seems unlikely that the higher order nonlinear terms omitted from the analysis could be large enough to influence the calculated results significantly, although a more careful treatment of mode self-interaction and its associated frequency shifts[5] may be necessary.

As a final observation we remark that, when there is appreciable inharmonicity in the string mode frequencies, whether from string stiffness or the complexity of the bridge admittance or other causes, it is possible to have much more complex behavior than we have studied here. This can be seen explicitly in the case of mode 3 which, from (12) or (23), can be driven by mode 1 at a frequency $3\omega_1$ or by modes 1 and 2 in combination at a frequency $2\omega_2$-ω_1. If these frequencies are not the same, then beatlike behavior ensues. This was easily observed in our mode 3 experiment by fixing a mass of a few tenths of a gram to the midpoint of the string so as to lower the frequency of mode 1 while leaving the mode 2 frequency unchanged.

The nonlinearities we have discussed also serve to couple in the same ways the modes that are actually excited on the string, so that they exchange energy and, if they are not ideally harmonic, fluctuate in amplitude.

VI. CONCLUSIONS

We have examined theoretically the nonlinear generation of missing modes on vibrating strings and have confirmed the predictions of the theory at least semi-quantitatively by experiment. The phenomenon has been shown to be confined to situations in which at least one of the bridges supporting the string has a nonzero mechanical admittance. This is, of course, the situation of practical importance in musical instruments.

Two different situations emerge from the analysis. If the nonrigid bridge has zero admittance parallel to the string length and if the string is fixed simply to it, then the only nonlinear processes tending to mix modes or to generate missing modes are of third order. On the other hand, if the bridge is a nonrigid support over which the string passes at an angle, as in many musical instruments, then there is an additional second-order nonlinearity which provides driving forces at frequencies that are twice those of any modes present on the string.

These nonlinearities, as well as generating appreciable amplitudes for modes initially not excited on the string, also serve to couple modes that are excited. In the practically important case in which mode frequencies are not in exact harmonic relationship, these couplings contribute fluctuations in the amplitudes of all the modes which undoubtedly influence to some extent the auditory perception of the sound produced.

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