with $H(t, q, p) = t^{n-1} \log(q^{n-1}/p) - kt^n q$. Because this is a non-autonomous system, with t appearing explicitly, the Hamiltonian H is *not* a constant of motion. However, (6) is invariant under the one-parameter group of scaling symmetries $r \to \mu r$, $f \to \mu^{-1} f$, so by introducing the new scale-invariant independent variable $y = ft$, and the dependent variables $v = v(y) = -\log t$, $w = \mathrm{d}v/\mathrm{d}y$, it reduces to a first-order equation for $w(y)$:

$$\frac{\mathrm{d}w}{\mathrm{d}y} = y(2n - ky)w^3 + (3n - 2ky)w^2 + ((n-1)/y - k)w.$$

Unfortunately, it is not possible to reduce this to a quadrature, since the action (7) is *not* invariant under scaling, unless $k = 0$ when the general solution to (6) is $f(t) = (At^n + B)^{1/n}$ (with A, B arbitrary constants).

Other important methods for ODEs include the Painlevé analysis of movable singularities (Kruskal & Clarkson, 1992), and asymptotic expansions around regular or irregular fixed singular points (Wasow, 1965; Tovbis, 1994).

ANDREW HONE

See also **Chaotic dynamics; Constants of motion and conservation laws; Euler–Lagrange equations; Extremum principles; Hamiltonian systems; Integrability; Painlevé analysis; Partial differential equations, nonlinear; Riccati equations**

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ORGANIZING CENTERS

See **Spiral waves**

OSCILLATOR, CLASSICAL NONLINEAR

See **Damped-driven anharmonic oscillator**

OVERTONES

When a tonal sound, such as a note played on a flute, a human vowel sound, or a bell, is analyzed in the frequency domain by applying a Fourier transform to the acoustic pressure waveform, the spectrum consists of a large number of sharp lines. The component of lowest frequency is termed the "fundamental", and the others are "upper partials" or "overtones." If the frequencies of the overtones are all exact integer multiples of the frequency of the fundamental, then they are termed "harmonics." The partial with frequency $f_n = nf_1$, where f_1 is the frequency of the fundamental, is the nth harmonic, so that the fundamental is the first harmonic.

A one-dimensional simple harmonic oscillator, or linear oscillator, in which the restoring force is proportional to displacement y from the equilibrium position, obeys the equation

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -ky, \qquad (1)$$

where m is the mass of the moving particle and k is the restoring force constant. A damping term can also be included, but this need not concern us here. Such an oscillator vibrates with a single frequency $f_1 = (1/2\pi)(k/m)^{1/2}$ that is independent of oscillation amplitude. It is useful to think of this oscillator in terms of its potential energy function, which is quadratic as shown in Figure 1(a). In real oscillators, the restoring force is not linear for large displacements, but nonlinear so that

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -ky(1 + \alpha_1 y + \alpha_2 y^2 + \cdots), \qquad (2)$$

where the α_n are constants. The energy curve then has a distorted parabolic form such as that shown as an example in Figure 1(b). In the absence of damping, the total energy must remain constant so that the magnitude of the velocity is a simple function of the displacement, and the motion repeats cyclically. This means that the spectrum of such a nonlinear oscillator consists of exact phase-locked harmonics of the fundamental frequency, though this fundamental frequency depends upon the amplitude of the motion. For a reason that is derived from molecular physics, as discussed below, such a nonlinear oscillator is often confusingly called an "anharmonic oscillator."

A thin taut string of length L ideally obeys an equation of the form

$$m\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}, \qquad (3)$$

where x measures length along the string, m is the mass per unit length, and T is the string tension.

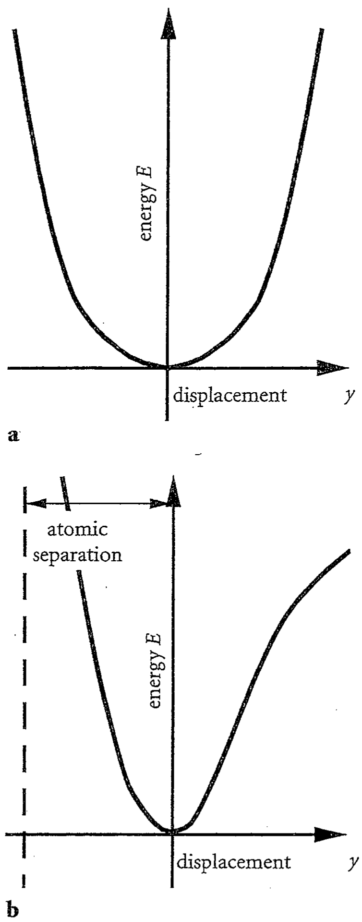

Figure 1. (a) Potential energy curve for a simple harmonic oscillator. (b) Potential energy curve for a typical nonlinear oscillator such as a diatomic molecule.

If its ends are rigidly fixed, then the mode frequencies are exact harmonics of the fundamental so that $f_n = (n/2L)(T/m)^{1/2}$. It is thus a multimode harmonic oscillator. The nonlinear frictional action of the bow on a violin reinforces the harmonicity of the modes and locks them into rigid phase relationship (Fletcher, 1999). Something very similar happens with wind instruments, which also have precisely harmonic spectra.

A thin stiff bar, on the other hand, obeys an equation of the form

$$m\frac{\partial^2 y}{\partial t^2} = K\frac{\partial^4 y}{\partial x^4}, \qquad (4)$$

where K is the elastic stiffness. If the ends are free or rigidly clamped, then the mode frequencies are approximately $f_n \approx \frac{4}{9}(n+\frac{1}{2})^2 f_1$, and the overtones are very far from being harmonically related. Such an oscillator might be termed "inharmonic." The modes of a three-dimensional object such as a bell are even more complex (Fletcher & Rossing, 1998).

While sustained-tone musical instruments depend upon the nonlinearity of the active generator for their operation (bow, reed, or lip air-flow), the linear resonator (string or air column) determines the oscillation frequency, so that the pitch is nearly independent of loudness, and only the relative amplitudes of the harmonics change (Fletcher, 1999). Some Chinese opera gongs, however, make a virtue of nonlinearity so that, after an impulsive excitation, the pitch either rises or falls dramatically as the vibration dies away (Fletcher, 1985). The frequencies and relative intensities of upper partials determine the tone quality of a musical sound and have dictated the development of musical scales and harmonies (Sethares, 1998).

The human auditory system itself has some nonlinear aspects (Zwicker & Fastl, 1999), and, as in any forced nonlinear oscillator, these lead to the generation of harmonics ("harmonic distortion") and of multiple sum and difference tones ("intermodulation distortion"). In the ear these are chiefly apparent in the generation of the difference tone $|f_1 - f_2|$ when loud tones of frequencies f_1 and f_2 are heard simultaneously.

Optical absorption and emission spectra have many similarities to acoustic phenomena (Herzberg, 1950; Harmony, 1989). Diatomic molecules, for example, have interatomic potentials of the form shown in Figure 1(b) and thus constitute nonlinear oscillators. If the interatomic potentials were simply parabolic, as in Figure 1(a), then the quantum energy levels would have the form $E_n = (n+\frac{1}{2})h\nu$, where h is Planck's constant and ν is the classical vibration frequency labeled f_1 above. The wave functions describing the atomic vibration would then be either symmetric or antisymmetric, and the selection rule would dictate that n could change only by ± 1. There would thus be only a single absorption band consisting of the vibrational transition $0 \rightarrow 1$ surrounded by the allowed rotational transition lines.

For a more realistic model of the interatomic potential, as in Figure 1(b), the energy levels can be written as

$$E_n = (n+\tfrac{1}{2})h\nu[1 + \beta_1(n+\tfrac{1}{2}) + \beta_2(n+\tfrac{1}{2})^2 + \cdots], \qquad (5)$$

where β_n are usually called the "coefficients of anharmonicity" and β_1 is always negative in practice. The asymmetry of the potential also relaxes the selection rule so that in addition to the strong allowed absorption transition $0 \rightarrow 1$, there are much weaker transitions from $n = 0$ to higher levels. The absorption bands associated with these transitions have frequencies that are in approximate, but not exact, harmonic relationship to the fundamental, and are called "overtone bands."

Although the quantum treatment of a nonlinear oscillator may seem to conflict with the classical treatment, and the term *anharmonic* certainly suggests this, there is not really any disagreement. The infrared spectrum is derived from transitions between two levels of different energies, and therefore different classical amplitudes, and the classical frequency depends upon amplitude, as for the Chinese opera gong.

NEVILLE FLETCHER

See also **Damped-driven anharmonic oscillator; Harmonic generation; Molecular dynamics; Nonlinearity, definition of; Ordinary differential equations, nonlinear; Partial differential equations, nonlinear; Spectral analysis**

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