HOKKAIDO Diagrams

A Geometrical Representation of Permutations

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During a recent study leave at Hokkaido University in Japan, one of us (NHF) ate lunch regularly with a party of graduate students in the Physics Departmental common room. A particular delicacy on these occasions was the soy bean cookies (manju) which had to be bought from the cafeteria, and the party decided that there were several small tasks associated with their purchase which should be distributed at random amongst its members. These tasks included collecting the orders, fetching the cookies and paying various amounts towards their purchase. The novel method used to make this random choice can be formalized into four rules:

1. Draw as many parallel lines on a piece of paper as there are members of the party. In the figures that follow, these are vertical.
2. Each member inserts an arbitrary number of cross-links between nearest-neighbour lines.
3. No neighbouring crosslinks can occur at the same horizontal level.

In a different version cross-links could join any pair of lines but we restrict ourselves here to links between nearest neighbours. An example for a party of three members is given in Figure 1. The three tasks X, Y and Z are put at the top of each line.

4. Each member A, B and C of the party starts at the bottom of his line and follows it upwards. When he meets a link he moves to the neighbouring line and so on until he reaches a task at the top. If this is done in Figure 1 it will be found that A gets task Y, B gets task X and C gets task Z.

Each member of the party contributes in a "democratic" way by having the opportunity of adding as many links as he likes.

We are going to call the diagrams "Hokkaido diagrams". The first result that we can deduce from the rules is that each member can get only one task. This follows from rule 4 by considering the simple diagram in Figure 2. The only link present allows member A to move from the left-hand line to the middle line and simultaneously member B moves from the middle line to the left-hand line. The link interchanges or transposes the members A and B on their two lines and thus each member's line will end on a single task. This interchange can be more readily seen by putting arrows

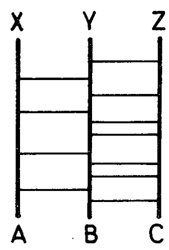

Fig. 1

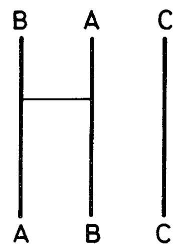

Fig. 2

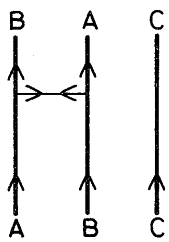

Fig. 3

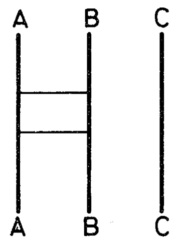

Fig. 4

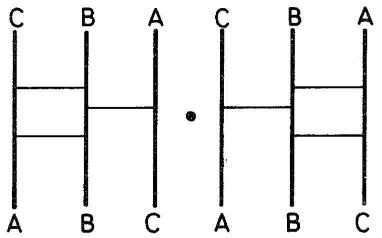

Fig. 5

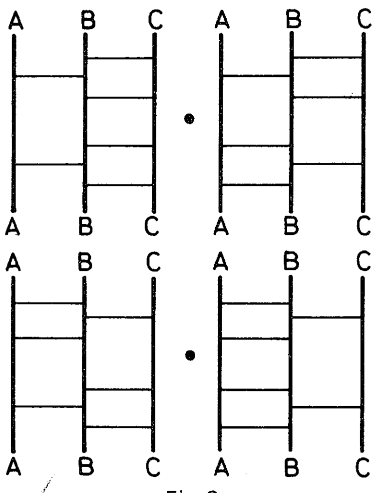

Fig. 6

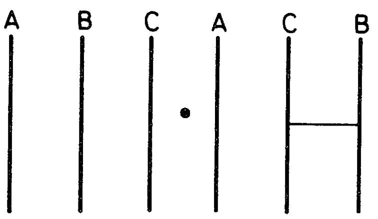

Fig. 7

on the diagram as in Figure 3, indicating the direction taken by a member of the party in reaching his task; a link carries two opposed arrows. Since from rule 1 there are only as many lines as there are members of the party, the pattern of the symbols A, B, C . . . attached to the top of the lines is then one of the permutations of the symbols A, B, C . . . at the bottom. Thus a Hokkaido diagram is a geometrical representation of one of the permutations of the permutation group of degree equal to the number of members in the party. Figure 2 is the permutation BAC of the three members ABC. We notice that the diagrams are "time-ordered" because a member starts on his line at the bottom of the diagram and ends at the top at a later time; time runs upwards in the diagrams. If we wish we can talk about "before" and "after" a diagram meaning respectively below and above geometrically.

There is an unlimited number of Hokkaido diagrams for each permutation because each member of the party can insert any number of links. At first sight this means that a Hokkaido diagram can be very complicated but it can be simplified by reducing the number of links while preserving the permutation; we will call this the "reduction" of the diagram. One way in which this reduction can be done is by successive application of rule 4. Consider the diagram in Figure 4. Application of rule 4 shows that the interchange of A and B by the first (lower) link is reversed by the second (upper) link and the consequence is that A is left with the task immediately above him. A more complicated example is given in Figures 5 and 6. Figure 5 shows the two diagrams in which A and C are interchanged. In Figure 6 we give the four ways in which these two diagrams can be combined to leave the pattern at the top the same as at the bottom. We note that the first and fourth diagrams in Figure 6 can be reduced by three successive applications of Figure 4 but the second and third diagrams cannot be so reduced. We now see that the use of the rules allows us to reduce a diagram to minimal diagram which is defined as that diagram which gives the permutation in question with the least number of links. There is not

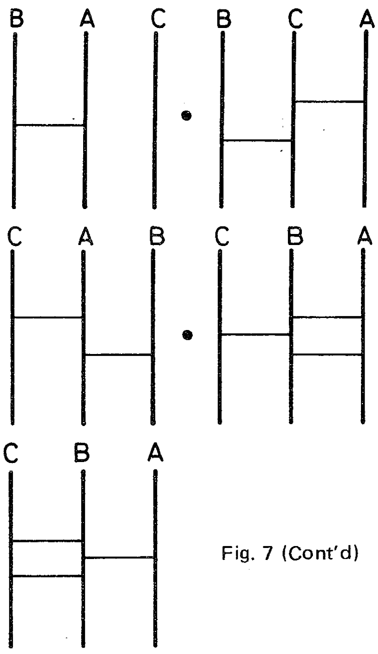

Fig. 7 (Cont'd)

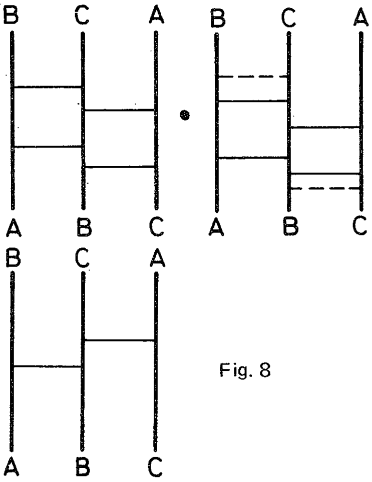

Fig. 8

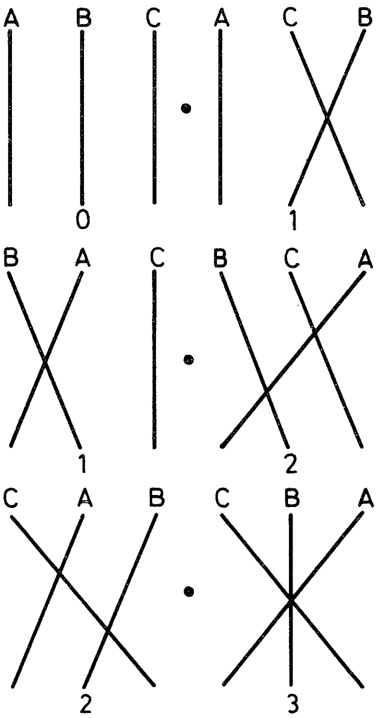

Fig. 9

always a unique minimal diagram; we define the number of minimal diagrams as the the minimal degeneracy. We now recognize that Figure 5 shows the minimal diagrams for the interchange of A and C and the minimal degeneracy is two.

We can now give in Figure 7 the minimal diagrams for all permutations of a party of three members; these permutations form the symmetric group of degree three, P_3. In each diagram of figure 7 the lines are assumed to be labelled A, B, C at the bottom.

The reduction to a minimal diagram can sometimes be made by adding intermediate lines. We give an example in Figure 8. Suppose that in reduction we had reached the first diagram in Figure 8. Since two interchanges of B and C leave the permutation unchanged, we do this in the second diagram by adding two links, shown dotted, one below and one above and the final reduction to the third and minimal diagram is achieved by twice using the result of Figure 4.

These Hokkaido diagrams should be contrasted with the representations in terms of twisted strings or crossovers. (Heine, 1960, p.50). We give these latter diagrams in Figure 9; each diagram has underneath it the parity of the permutation, which is the number of consecutive interchanges needed to get from the original arrangement (here, ABC) to the final arrangement.

Comparing Figures 7 and 9 we see that the parity of a permutation is the number of intersections of neighbouring lines in the diagrams of Figure 9 and also the number of links in the minimal Hokkaido diagrams. Using our language, the twisted string diagrams are non-degenerate.

The symmetric group P_3 is isomorphic to the group representing the symmetries of a plane equilateral triangle (Heine, 1960, p.22). The identity operation and the two operations, rotation by $2\pi/3$ and $4\pi/3$ form a subgroup of P_3. The corresponding Hokkaido diagrams are ABC, BCA and CAB in Figure 7, referring to them by the symbols at the top of their diagrams. The remaining three diagrams of Figure 7 represent the rotations through π about the three apices of the triangle.

In Figure 10 we give the Hokkaido diagrams for the 4! permutations of four objects; these make up the symmetric group P_4. The symbols underneath each diagram refer to the group generators. These allow a compact description of diagrams.

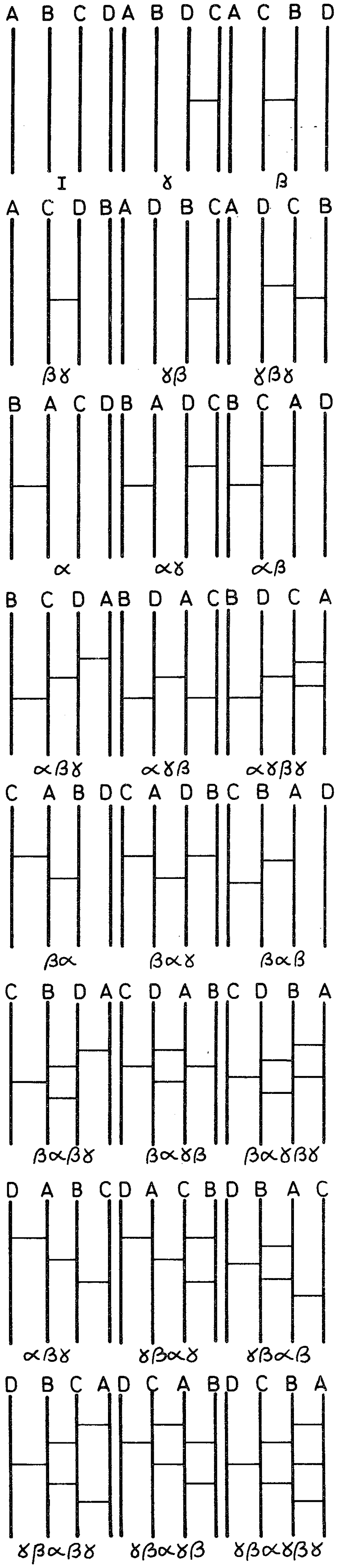

Fig. 10 — 4 sets of six diagrams

We define the identity I and the generators a, β for P_3 in Figure 11 (Carmichael, 1937, p.34). The defining relations are

$$a\,a = \beta\,\beta = I \qquad (1)$$

$$a\beta a\beta a\beta = I \qquad (2)$$

In using equations (1) we must remember that the diagrams are time-ordered; our convention is that the first generator in a word or a relation is at the earlier time. Thus $a\,a = I$ represents Figure 3. The diagrams in Figure 7 are described by

$$I, \beta, a, a\beta, \beta a, \beta a\beta = a\beta a$$

For the group P_4 we choose the generators a, β, γ, as in Figure 12. The relations are

$$aa = \beta\beta = \gamma\gamma = I, \qquad (3)$$

$$a\gamma\beta\gamma\beta\gamma\beta a = I, \qquad (4)$$

$$\gamma a\beta a\beta a\beta\gamma = I. \qquad (5)$$

We must also note the three subsidiary relations

$$a\beta \neq \beta a; \qquad (6)$$

$$\beta\gamma \neq \gamma\beta, \qquad (7)$$

$$a\gamma = \gamma a. \qquad (8)$$

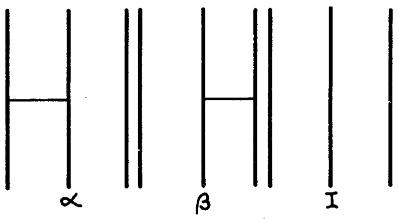

Fig. 11

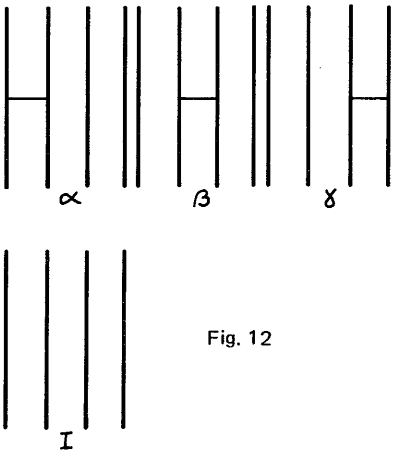

Fig. 12

Returning to Figure 10, we note that the diagrams in the first set are given by adding one vertical line to the diagrams in Figure 7, those in the second set are given by premultiplying those in the first row by a, those in the third row by premultiplying those in the second by β and those in the third by γ. The minimal diagram with the largest number of links is that for the complete reversal DCBA namely $\gamma\beta a\gamma\beta\gamma$. This technique can be readily extended to the symmetric groups of any degree n and we see that for P_n, the largest number of links is given by the sum of the first (n–1) natural numbers, namely ½n(n–1).

Acknowledgement

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References

Heine, V. *Group Theory in Quantum Mechanics,* Pergamon London (1960).

Carmichael R. C., *Groups of Finite Order,* Ginn., N.Y., (1937).

Conferences

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Organized by the Science, Education and Management Division of the Institution of Electrical Engineers in Association with The Institute of Physics and the Institution of Electronic and Radio Engineers, the 1978 conference in gas discharges will be held on 11 – 15 September 1978.

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