The Centre of Mass

- The coordinates of the centre of mass are
  \[ x_{CM} = \frac{\sum m_i x_i}{M} \]
  \[ y_{CM} = \frac{\sum m_i y_i}{M} \]
  \[ z_{CM} = \frac{\sum m_i z_i}{M} \]

  \( M \) is the total mass of the system

- There is a special point in a system or object, called the centre of mass, that moves as if all of the mass of the system is concentrated at that point.

- The system will move as if an external force were applied to a single particle of mass \( M \) located at the centre of mass.
Extended Object

- Similar analysis can be done for an extended object
- Consider the extended object as a system containing a large number of particles
- Since particle separation is very small, it can be considered to have a constant mass distribution
Centre of Mass, position

- The centre of mass in three dimensions can be located by its position vector, $\vec{r}_{CM}$.
- For a system of particles,
  \[ \vec{r}_{CM} = \frac{1}{M} \sum_i m_i \vec{r}_i \]
  - $\vec{r}_i$ is the position of the $i^{th}$ particle, defined by
    \[ \vec{r}_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k} \]
- For an extended object,
  \[ \vec{r}_{CM} = \frac{1}{M} \int \vec{r} \, dm \]
Centre of Mass, Symmetric Object

- The centre of mass of any symmetric object lies on an axis of symmetry and on any plane of symmetry
  - If the object has uniform density

Finding CoM for Irregularly Shaped Object

- Suspend the object from one point
- Then suspend from another point
- The intersection of the lines A-B and C-D is the centre of mass
Centre of Gravity

- Each small mass element of an extended object is acted upon by the gravitational force.
- The net effect of all these forces is equivalent to the effect of a single force $M\vec{g}$ acting through a point called the **centre of gravity**.
  - If $\vec{g}$ is constant over the mass distribution, the centre of gravity coincides with the centre of mass.
Example – the Centre of Mass of a Rod

Show that the centre of mass of a rod of mass $M$ and length $L$ lies midway between its end, assuming the rod has a uniform mass per unit length (linear density)

For an extended object, definition of $C$ of $M$ is
\[ x_{CM} = \frac{1}{M} \int x \, dm \]

Mass per unit length $\lambda = \frac{M}{L}$ for a uniform rod

or, in general $dm = \lambda \, dx \Rightarrow x_{CM} = \frac{1}{M} \int_0^L x \, \lambda \, dx$

For a uniform linear mass density, i.e. $\lambda$ independent of $x$

\[ x_{CM} = \frac{1}{M} \lambda \int_0^L x \, dx = \frac{1}{M}(\frac{M}{L})(\frac{L^2}{2}) = \frac{L}{2} \]
Motion of a System of Particles

- Assume the total mass, $M$, of the system remains constant
- We can describe the motion of the system in terms of the velocity and acceleration of the centre of mass of the system
- We can also describe the momentum of the system and Newton’s Second Law for the system

Velocity and Momentum

- The velocity of the centre of mass of a system of particles is
  \[
  \vec{v}_{\text{CM}} = \frac{d\vec{r}_{\text{CM}}}{dt} = \frac{1}{M} \sum_i m_i \vec{v}_i
  \]
- The momentum can be expressed as
  \[
  M\vec{v}_{\text{CM}} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{\text{tot}}
  \]
- The total linear momentum of the system equals the total mass multiplied by the velocity of the centre of mass
Acceleration of the Centre of Mass

- The acceleration of the centre of mass can be found by differentiating the velocity with respect to time

\[ \vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M} \sum m_i \vec{a}_i \]

Forces In a System of Particles

- The acceleration can be related to a force

\[ M\vec{a}_{CM} = \sum_i \vec{F}_i \]

- If we sum over all the internal forces, they cancel in pairs and the net force on the system is caused only by the external forces
Newton’s Second Law for a System of Particles

- Since the only forces are external, the net external force equals the total mass of the system multiplied by the acceleration of the centre of mass:

\[ \sum \vec{F}_{ext} = M \vec{a}_{CM} \]

- The centre of mass of a system of particles of combined mass \( M \) moves like an equivalent particle of mass \( M \) would move under the influence of the net external force on the system.
Impulse and Momentum of a System of Particles

- The impulse imparted to the system by external forces is

\[ \vec{I} = \int \sum \vec{F}_{\text{ext}} \, dt = M \int d\vec{v}_{CM} = \Delta \vec{p}_{tot} \]

- The total linear momentum of a system of particles is conserved if no net external force is acting on the system

\[ M \vec{v}_{CM} = \vec{p}_{tot} = \text{constant} \quad \text{when} \quad \sum \vec{F}_{\text{ext}} = 0 \]
Motion of the Center of Mass

- A projectile is fired into the air and suddenly explodes.
- With no explosion, the projectile would follow the dotted line.
- After the explosion, the center of mass of the fragments still follows the dotted line, the same parabolic path the projectile would have followed with no explosion.
Deformable Systems

- To analyze the motion of a deformable system, use Conservation of Energy and the Impulse-Momentum Theorem

\[ \Delta E_{\text{system}} = \sum T \rightarrow \Delta K + \Delta U = 0 \]

\[ \vec{I} = \Delta \vec{p}_{\text{tot}} \rightarrow \int \vec{F}_{\text{ext}} \, dt = m \Delta \vec{v} \]

- If the force is constant, the integral can be easily evaluated

Push on left block, it moves to right, spring compresses

- At any given time, the blocks are generally moving with different velocities
- The blocks oscillate back and forth with respect to the centre of mass
Rocket Propulsion

- The operation of a rocket depends upon the law of conservation of linear momentum as applied to a system of particles, where the system is the rocket plus its ejected fuel.

- The initial mass of the rocket plus all its fuel is $M + \Delta m$ at time $t_i$ and speed $v$.

- The initial momentum of the system is

$$\vec{p}_i = (M + \Delta m)\vec{v} \quad (a)$$

Rocket Propulsion
At some time \( t + \Delta t \), the rocket’s mass has been reduced to \( M \) and an amount of fuel, \( \Delta m \) has been ejected.

The rocket’s speed has increased by \( \Delta v \).

Because the gases are given some momentum when they are ejected out of the engine, the rocket receives a compensating momentum in the opposite direction.

Therefore, the rocket is accelerated as a result of the “push” from the exhaust gases.

In free space, the centre of mass of the system (rocket plus expelled gases) moves uniformly, independent of the propulsion process.
Conservation of momentum, \((M + \Delta m)v = M(v + \Delta v) + \Delta m(v - v_e)\)

EXHAUST SPEED - speed of ejected fuel, \(v_e\), relative to \(v = 0\) \((v - v_e)\)

i.e., \(Mv + \Delta mv = Mv + M\Delta v + \Delta mv - \Delta mv_e \Rightarrow M\Delta v = \Delta mv_e\)

Take limit as \(\Delta t \to 0\) \(\Rightarrow Mdv = v_e dm = v_e dM\)

\(\Rightarrow \int dv = -v_e \int dM/M\)

\(\Rightarrow V_f - V_i = v_e \ln(M_i/M_f)\)
The basic equation for rocket propulsion is

\[ v_f - v_i = v_e \ln \left( \frac{M_i}{M_f} \right) \]

- The increase in rocket speed is proportional to the speed of the escape gases \(v_e\) - so, the exhaust speed should be very high.
- The increase in rocket speed is also proportional to the natural log of the ratio \(M_i/M_f\).
  - So, the ratio should be as high as possible, meaning the mass of the rocket should be as small as possible and it should carry as much fuel as possible.
- The thrust on the rocket is the force exerted on it by the ejected exhaust gases

\[
\text{thrust} = M \frac{dv}{dt} = v_e \frac{dM}{dt}
\]

- The thrust increases as the exhaust speed increases.
- The thrust increases as the rate of change of mass increases.
  - The rate of change of the mass is called the burn rate.
Rigid Object

- A rigid object is one that is nondeformable
  - The relative locations of all particles making up the object remain constant
  - All real objects are deformable to some extent, but the rigid object model is very useful in many situations where the deformation is negligible
- This simplification allows analysis of the motion of an extended object
Angular Position

- Axis of rotation is the center of the disc
- Choose a fixed reference line
- Point P is at a fixed distance r from the origin
  - A small element of the disc can be modeled as a particle at P

- Point P will rotate about the origin in a circle of radius r
- Every particle on the disc undergoes circular motion about the origin, O
- Polar coordinates are convenient to use to represent the position of P (or any other point)
- P is located at \((r, \theta)\) where r is the distance from the origin to P and \(\theta\) is the measured counterclockwise from the reference line
- As the particle moves, the only coordinate that changes is $\theta$.
- As the particle moves through $\theta$, it moves through an arc length $s$.
- The arc length and $r$ are related:
  - $s = \theta r$

$\Rightarrow \theta = s/r$, the definition of a radian.
Radian

\[ \theta = \frac{s}{r} \]

- \( \theta \) is a pure number, but commonly is given the artificial unit, radian
- One radian is the angle subtended by an arc length equal to the radius of the arc
- Whenever using rotational equations, you must use angles expressed in radians
  
  - Comparing degrees and radians
    
    \[ 1 \text{ rad} = \frac{360^\circ}{2\pi} = 57.3^\circ \]
  - Converting from degrees to radians
    
    \[ \theta (rad) = \frac{\pi}{180^\circ} \theta (degrees) \]
Angular Position, final

● We can associate the angle $\theta$ with the entire rigid object as well as with an individual particle
  ● Remember every particle on the object rotates through the same angle

● The angular position of the rigid object is the angle $\theta$ between the reference line on the object and the fixed reference line in space
  ● The fixed reference line in space is often the $x$-axis
Angular Displacement

- The *angular displacement* is defined as the angle the object rotates through during some time interval.

\[ \Delta \theta = \theta_f - \theta_i \]

- This is the angle that the reference line of length \( r \) sweeps out.
The average angular speed, \( \omega_{\text{avg}} \), of a rotating rigid object is the ratio of the angular displacement to the time interval

\[
\omega_{\text{avg}} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta \theta}{\Delta t}
\]

The instantaneous angular speed is defined as the limit of the average speed as the time interval approaches zero

\[
\omega \equiv \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t} = \frac{d \theta}{dt}
\]

cf. \( v = \frac{ds}{dt} = rd\theta/dt \) \( \Rightarrow v = \omega r \)

Units of angular speed are radians/sec  
Also called angular frequency
- rad/s or \( s^{-1} \) since radians have no dimensions \( (s^{-1} = \text{Hz}) \)

Angular speed will be positive if \( \theta \) is increasing (counterclockwise)

Angular speed will be negative if \( \theta \) is decreasing (clockwise)
Angular Acceleration

- The average angular acceleration, $\alpha$, of an object is defined as the ratio of the change in the angular speed to the time it takes for the object to undergo the change:

$$\alpha_{\text{avg}} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{\Delta \omega}{\Delta t}$$

- The instantaneous angular acceleration is defined as the limit of the average angular acceleration as the time goes to 0

$$\alpha \equiv \lim_{\Delta t \to 0} \frac{\Delta \omega}{\Delta t} = \frac{d\omega}{dt} \quad \text{cf. } a = \frac{dv}{dt}$$

- Units of angular acceleration are rad/s$^2$ or s$^{-2}$ since radians have no dimensions

- Angular acceleration will be positive if an object rotating counterclockwise is speeding up

- Angular acceleration will also be positive if an object rotating clockwise is slowing down