Truncated eigenvalue equation and long wavelength behavior of lattice gauge theory

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We review our new method, which might be the most direct and efficient way for approaching the continuum physics from Hamiltonian lattice gauge theory. It consists of solving the eigenvalue equation with a truncation scheme preserving the continuum limit. The efficiency has been confirmed by the observations of the scaling behaviors for the long wavelength vacuum wave functions and mass gaps in (2+1)-dimensional models and (1+1)-dimensional \( \sigma \) model even at very low truncation orders. Most of these results show rapid convergence to the available Monte Carlo data, ensuring the reliability of our method.

1. INTRODUCTION

The main purpose of our work is to approach the scaling region and extract physical results by analytic calculations. For a lattice calculation, a basic requirement is that for weak enough coupling, the dimensionless quantities should satisfy the scaling law, predicted by renormalization group equation. For SU(\( N_c \)) gauge theories in 3 dimensions, superrenormalizability and dimensional analysis tell us that the dimensionless masses \( aM \) should scale as

\[
\frac{aM}{g^2} \rightarrow \frac{M}{e^2}.
\]

For (2+1)-dimensional compact U(1) and (3+1)-dimensional non-abelian gauge theories, \( aM \) should scale exponentially as

\[
aM \rightarrow \exp(-b/g^2).
\]

If the calculated \( M \) data converge to a stable value, we can get an estimate for the mass.

There have been various analytic methods available in the literature (for a review see [1]). The main difficulty of the conventional methods (e.g. strong coupling expansion) is that they converge very slowly and very higher order 1/\( g^2 \) calculations are required to extend the results to the intermediate coupling region. Unfortunately, high order calculations are difficult in practice.

Recently, we proposed a new method [2–5] for Hamiltonian lattice gauge theory. This method consists of solving the eigenvalue equation with a suitable truncation scheme preserving the continuum limit. Even at low order truncation, clear scaling windows for the physical quantities in most cases have been established, and the results are in perfect agreement with the available Monte Carlo data. Here we review only the work on U(1)\(_3\), SU(2)\(_3\) and 2 dimensional \( \sigma \) model, while that for SU(3) has been summarized in [6].

2. THE METHOD

The Schrödinger equation \( H|\Omega\rangle = \epsilon_{\Omega}|\Omega\rangle \) on the Hamiltonian lattice for the ground state

\[
|\Omega\rangle = \exp[R(U)]|0\rangle
\]

and vacuum energy \( \epsilon_{\Omega} \) can be reformulated as

\[
\sum_i \{ [E_i, [E_i, R(U)]] + [E_i, R(U)][E_i, R(U)] \}
\]

\[
- \frac{2}{g^2} \sum_p \text{tr}(U_p + U_p^\dagger) = \frac{2a}{g^2} \epsilon_{\Omega}.
\]
To solve this equation, let us write \( R(U) \) in order of graphs \( G_{n,i} \), i.e., \( R(U) = \sum_n R_n(U) = \sum_{n,i} C_{n,i} G_{n,i}(U) \). Substituting it to (4), we have the \( N \)th order truncated eigenvalue equation

\[
\sum_i \{ E_i, [E_i, \sum_n R_n(U)] \} + \sum_{n_1+n_2 \leq N} \{ E_i, R_{n_1}(U) \} [E_i, R_{n_2}(U)]
\]

\[
- \frac{2}{g_4^2} \sum_p \text{tr}(U_p + U'_p) = \frac{2a}{g^2} \epsilon_{\Omega}.
\] (5)

By taking the coefficients of the graphs \( G_{n,i} \) in this equation to zero, we obtain a set of non-linear algebra equations, from which \( C_{n,i} \) are determined. The similar method applies to the eigenvalue equation for the mass and its wave function [5]. Therefore, solving lattice field theory is reduced to solving the algebra equations.

The lowest order graph is quite simple: \( R_1(U) = C_{1,1}(U_p + \text{h.c.}) \) The first term in (5) doesn’t generate new graphs, but the second term does, i.e.

\[
[E_i, G_{n_1}(U)] \in R_n(U) + \text{lower orders},
\]

\[
[E_i, G_{n_1}(U)] [E_i, G_{n_2}(U)] \in R_{n_1+n_2}(U)
\]

+ lower orders. (6)

Two questions arise:

1) Should all the new graphs generated by the second term in (5) be taken as independent graphs of order \( n_1 + n_2 \)? For abelian gauge theories, the answer is yes. For non-abelian gauge theories, because of the uni-modular conditions [4–6,13], there is a mixing problem not only for the graphs of the same order, but also for graphs of different orders. The classification for independent graphs is particularly complicate for SU(3).

2) For \( n_1 + n_2 > N \), should we keep the lower order graphs in \( [E_i, G_{n_1}(U)] [E_i, G_{n_2}(U)] \)? To preserve the correct limit, at \( N \)th order truncation, one should to DROP all these graphs. This is the essential feature of our truncation scheme, which differs sufficiently from the scheme in [7].

There have also been some other truncation schemes proposed in [8,9]. One of their major problems is the violation of the long wavelength structure or continuum limit of the equation, and consequently the violation of the scaling law (1) or (2) for the physical quantities.

Let’s see further why the equation (5) should be truncated in the way suggested at point 2). The continuum limit of a graph \( G_{n,i}(U) \) is

\[
G_{n,i}(U) = e^2 a^4 [A_{n,i} \text{ tr}(\mathcal{F}^2)] + a^2 B_{n,i} \text{ tr}(\mathcal{D}\mathcal{F})^2 \] + ... (7)

with \( \mathcal{F} \) the field strength tensor and \( \mathcal{D} \) the covariant derivative. It has been generally proven [2] that in the continuum limit the second term of (5) term should behave as

\[
[E_i, G_{n_1}(U)] [E_i, G_{n_2}(U)] \propto a^6 \text{ Tr}(D\mathcal{F}_{\mu,\nu})^2. \] (8)

To preserve this correct limit, when the equation (5) is truncated to the \( N \)th order, all the graphs created by \( [E_i, G_{n_1}(U)] [E_i, G_{n_2}(U)] \) for \( n_1 + n_2 \leq N \) must be considered. On the other hand, all the graphs created by this term for \( n_1 + n_2 > N \) should be dropped, even there are lower order graphs. Otherwise the partial sum of the lower order graphs would make this term behave in a considerably different (wrong) way.

3. RESULTS

Once the coefficients \( C_{n,i} \) are obtained by solving (5), we can use (3) and (7) to compute the parameters \( \mu_0 \) and \( \mu_2 \) in the vacuum wave function for the long wavelength configurations \( U \) [10]

\[
\langle \Omega \rangle = \exp[-\mu_0 \int d^{D-1}x \text{ tr}\mathcal{F}^2 - \mu_2 \int d^{D-1}x \text{ tr}(\mathcal{D}\mathcal{F})^2]. \] (9)

The results for \( \mu_0 \) and \( \mu_2 \) in 3 dimensional SU(2) gauge theory are shown in Fig. 1. Empressively, nice scaling behavior is obtained even at \( N = 3 \), and the data for \( 4/g^2 > 4 \) are in good agreement with the Monte Carlo measurements [10]. The order \( N = 4 \) data are also included in
Figure 1. Parameters in the vacuum wave function of SU(2)\textsubscript{3} model. The dashed lines show the mean values for the Monte Carlo data.

Figure 2. The mass gap of the SU(2)\textsubscript{3} model from three different methods. The dashed lines show the continuum limit of the Monte Carlo data.

The most intriguing scaling law is the exponential scaling (2). Before investigating QCD in 3+1 dimensions, we would like to test our method in a (2+1)-dimensional compact U(1) model, which has many properties of the realistic theory. Here there is no ambiguity induced by uni-modular conditions in SU(N\textsubscript{c}) theories. Because the nature of the abelian group greatly simplifies the calculations, we can easily write a program at arbitrary orders.

As is mentioned above, for non-abelian gauge theories, the uni-modular conditions lead to the existence of different choices for independent graphs. In SU(2), because of trU\textsuperscript{T} = trU, all the disconnected graphs can be transformed into the connected ones, used as an independent set of the graphs. Our results in Figs. 1 and 2 are from such a choice, while the comparison between different choices (connected, disconnected [2] and inverse) has been made in [13]. Of course, one is free to choose arbitrary set of independent graphs. A criterion for a good choice is that it is convergent more rapidly to the continuum limit than other ones at lower order truncation. This has been obviously demonstrated for SU(3) [6,14].

Figure 3. Relevant quantity for μ\textsubscript{0} in the long wavelength vacuum state of compact U(1)\textsubscript{3}. The dashed line is the expected scaling law.
4. SUMMARY

The reason for the success of our method is that the truncated eigenvalue equation preserves the continuum limit. The results for (2+1)-dimensional models and (1+1)-dimensional $\sigma$ models are presented to support the efficiency and reliability of our method. In conclusion, the eigenvalue equation with a proper truncation scheme may be the most direct and efficient way for extracting the continuum physics.

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