20.5 Arithmetic Coding

We saw in the previous section that a perfect (entropy-bounded) coding scheme would use \( L_i = -\log_2 p_i \) bits to encode character \( i \) (in the range \( 1 \leq i \leq N_{ch} \)), if \( p_i \) is its probability of occurrence. Huffman coding gives a way of rounding the \( L_i \)'s to close integer values and constructing a code with those lengths. Arithmetic coding\(^1\), which we now discuss, actually does manage to encode characters using noninteger numbers of bits! It also provides a convenient way to output the result not as a stream of bits, but as a stream of symbols in any desired radix. This latter property is particularly useful if you want, e.g., to convert data from bytes (radix 256) to printable ASCII characters (radix 94), or to case-independent alphanumeric sequences containing only A-Z and 0-9 (radix 36).

In arithmetic coding, an input message of any length is represented as a real number \( R \) in the range \( 0 \leq R < 1 \). The longer the message, the more precision required of \( R \). This is best illustrated by an example, so let us return to the fictitious language, Vowelish, of the previous section. Recall that Vowelish has a 5 character alphabet (A, E, I, O, U), with occurrence probabilities 0.12, 0.42, 0.09, 0.30, and 0.07, respectively. Figure 20.5.1 shows how a message beginning “IOU” is encoded: The interval \([0, 1)\) is divided into segments corresponding to the 5 alphabetical characters; the length of a segment is the corresponding \( p_i \). We see that the first message character, “I”, narrows the range of \( R \) to \( 0.37 \leq R < 0.46 \). This interval is now subdivided into five subintervals, again with lengths proportional to the \( p_i \)'s. The second message character, “O”, narrows the range of \( R \) to \( 0.3763 \leq R < 0.4033 \). The “U” character further narrows the range to \( 0.37630 \leq R < 0.37819 \). Any value of \( R \) in this range can be sent as encoding “IOU”. In particular, the binary fraction .011000001 is in this range, so “IOU” can be sent in 9 bits. (Huffman coding took 10 bits for this example, see §20.4.)

Of course there is the problem of knowing when to stop decoding. The fraction .011000001 represents not simply “IOU,” but “IOU...” where the ellipses represent an infinite string of successor characters. To resolve this ambiguity, arithmetic coding generally assumes the existence of a special \( N_{ch} + 1 \)th character, EOM (end of message), which occurs only once at the end of the input. Since EOM has a low probability of occurrence, it gets allocated only a very tiny piece of the number line.

In the above example, we gave \( R \) as a binary fraction. We could just as well have output it in any other radix, e.g., base 94 or base 36, whatever is convenient for the anticipated storage or communication channel.

You might wonder how one deals with the seemingly incredible precision required of \( R \) for a long message. The answer is that \( R \) is never actually represented all at once. At any given stage we have upper and lower bounds for \( R \) represented as a finite number of digits in the output radix. As digits of the upper and lower bounds become identical, we can left-shift them away and bring in new digits at the low-significance end. The routines below have a parameter \( NWK \) for the number of working digits to keep around. This must be large enough to make the chance of an accidental degeneracy vanishingly small. (The routines signal if a degeneracy ever occurs.) Since the process of discarding old digits and bringing in new ones is performed identically on encoding and decoding, everything stays synchronized.
The routine \texttt{arcmak} constructs the cumulative frequency distribution table used to partition the interval at each stage. In the principal routine \texttt{arcode}, when an interval of size \(j\text{dif}\) is to be partitioned in the proportions of some \(n\) to some \(n\text{tot}\), say, then we must compute \((n\times\text{dif})/\text{tot}\). With integer arithmetic, the numerator is likely to overflow; and, unfortunately, an expression like \(\text{dif}/(\text{tot}/n)\) is not equivalent. In the implementation below, we resort to double precision floating arithmetic for this calculation. Not only is this inefficient, but different roundoff errors can (albeit very rarely) make different machines encode differently, though any one type of machine will decode exactly what it encoded, since identical roundoff errors occur in the two processes. For serious use, one needs to replace this floating calculation with an integer computation in a double register (not available to the \texttt{FORTRAN} programmer).

The internally set variable \texttt{minint}, which is the minimum allowed number of discrete steps between the upper and lower bounds, determines when new low-significance digits are added. \texttt{minint} must be large enough to provide resolution of all the input characters. That is, we must have \(p_i \times \text{minint} > 1\) for all \(i\). A value of \(100N_{ch}\), or \(1.1/\min p_i\), whichever is larger, is generally adequate. However, for safety, the routine below takes \texttt{minint} to be as large as possible, with the product \(\text{minint}\times\text{nrad}d\) just smaller than overflow. This results in some time inefficiency, and in a few unnecessary characters being output at the end of a message. You can
decrease minint if you want to live closer to the edge.

A final safety feature in arcmak is its refusal to believe zero values in the table nfreq; a 0 is treated as if it were a 1. If this were not done, the occurrence in a message of a single character whose nfreq entry is zero would result in scrambling the entire rest of the message. If you want to live dangerously, with a very slightly more efficient coding, you can delete the \( \max( , 1) \) operation.

SUBROUTINE arcmak(nfreq,nchh,nradd)
INTEGER nchh,nradd,nfreq(nchh),MC,NWK,MAXINT
PARAMETER (MC=512,NWK=20,MAXINT=2147483647)

Given a table nfreq(1:nchh) of the frequency of occurrence of nchh symbols, and given a desired output radix nradd, initialize the cumulative frequency table and other variables for arithmetic compression.

Parameters: MC is largest anticipated value of nchh; NWK is the number of working digits (see text); MAXINT is a large positive integer that does not overflow.

INTEGER j,jdif,minint,nc,nch,nrad,ncum,
* ncumfq(MC+2),ilob(NWK),iupb(NWK)
COMMON /arccom/ ncumfq,iupb,ilob,nch,nrad,minint,jdif,nc,ncum
SAVE /arccom/
if(nchh.gt.MC)pause 'MC too small in arcmak'
if(nradd.gt.256)pause 'nradd may not exceed 256 in arcmak'
minint=MAXINT/nradd
nch=nchh
nrad=nradd
ncumfq(1)=0
do:11
ncumfq(j)=ncumfq(j-1)+max(nfreq(j-1),1)
enddo:11
ncumfq(nch+2)=ncumfq(nch+1)+1
ncum=ncumfq(nch+2)
return
END

Individual characters in a message are coded or decoded by the routine arcode, which in turn uses the utility arcsam.

SUBROUTINE arcode(ich,code,lcode,lcd,isign)
INTEGER ich,isign,lc,lcode,MC,NWK
CHARACTER*1 code(lcode)
PARAMETER (MC=512,NWK=20)
C USES arcsam

Compress (isign = 1) or decompress (isign = -1) the single character ich into or out of the character array code(1:lcode), starting with byte code(lc) and (if necessary) incrementing lc so that, on return, lc points to the first unused byte in code. Note that this routine saves the result of previous calls until a new byte of code is produced, and only then increments lc. An initializing call with isign=0 is required for each different array code. The routine arcmak must have previously been called to initialize the common block /arccom/. A call with ich=nch (as set in arcmak) has the reserved meaning “end of message.”

INTEGER ihi,j,ja,jdif,jb,jl,k,m,minint,nc,nch,nrad,ilob(NWK),
* iupb(NWK),ncumfq(MC+2),ncum,JTRY
COMMON /arccom/ ncumfq,iupb,ilob,nch,nrad,minint,jdif,nc,ncum
SAVE /arccom/

The following statement function is used to calculate \((k+j)/m\) without overflow. Program efficiency can be improved by substituting an assembly language routine that does integer multiply to a double register.

\[ JTRY(j,k,m)=\text{int}(\text{dble}(k)*\text{dble}(j))/\text{dble}(m) \]
if (isign.eq.0) then
  jdif=nrad-1
do: j=NWK,1,-1
iupb(j)=nrad-1
ilob(j)=0
nc=j
if(jdif.gt.minint)return Initialization complete.
jdif=(jdif+1)*nrad-1
endo11
pause 'NWK too small in arcode'
else
if (isign.gt.0) then
  If encoding, check for valid input character.
  if((ich.gt.nch.or.ich.lt.0))pause 'bad ich in arcode'
else
  If decoding, locate the character ich by bisection.
ja=ichar(code(lcd))-ilob(nc)
do 12: j=nc+1,NWK
   ja=ja*nrad+(ichar(code(j+lcd-nc))-ilob(j))
endo12
ich=0
ihi=nch+1
1 if(ihi-ich.gt.1) then
   m=(ich+ihi)/2
   if (ja.ge.JTRY(jdif,ncumfq(m+1),ncum)) then
      ich=m
   else
      ihi=m
   endif
   goto 1
endif
if(ich.eq.nch)return Detected end of message.
endif
Following code is common for encoding and decoding. Convert character ich to a new subrange [ilob,iupb).
jh=JTRY(jdif,ncumfq(ich+2),ncum)
jl=JTRY(jdif,ncumfq(ich+1),ncum)
jdif=jh-jl
call arcsum(ilob,iupb,jh,NWK,nrad,nc)
call arcsum(ilob,ilob,jl,NWK,nrad,nc) How many leading digits to output
2 do 13: j=nc,NWK
   if(ich.ne.nch.and.iupb(j).ne.ilob(j))goto 2
   if(lcd.gt.lcode)pause 'lcode too small in arcode'
   if(isign.gt.0) code(lcd)=char(ilob(j))
lcd=lcd+1
endo13
return Ran out of message. Did someone forget to encode
   a terminating ncd?
3 if (jdif.lt.minint) then
   j=j+1
   jdif=jdif*nrad
goto 3
endif
if (nc-j.lt.1) pause 'NWK too small in arcode'
if(j.ne.0) then Shift them.
do 14: k=nc,NWK
   iupb(k-j)=iupb(k)
ilob(k-j)=ilob(k)
endo14
endif
nc=nc-j
ndo 15: k=NWK-j+1,NWK
   iupb(k)=0
   ilob(k)=0
endo15
return Normal return.
END
SUBROUTINE arcsum(iin, iout, ja, nwk, nrad, nc)
INTEGER ja, nc, nrad, nwk, iin(*), iout(*)
    Used by arcode. Add the integer ja to the radix nrad multiple-precision integer iin(nc..nwk).
    Return the result in iout(nc..nwk).
INTEGER j, jtmp, karry
    karry = 0
    do 1 j = nwk, nc+1, -1
        jtmp = ja
        ja = ja / nrad
        iout(j) = iin(j) + (jtmp - ja * nrad) + karry
        if (iout(j) .ge. nrad) then
            iout(j) = iout(j) - nrad
            karry = 1
        else
            karry = 0
        endif
    enddo
    iout(nc) = iin(nc) + ja + karry
    return
END

If radix-changing, rather than compression, is your primary aim (for example to convert an arbitrary file into printable characters) then you are of course free to set all the components of nfreq equal, say, to 1.

CITED REFERENCES AND FURTHER READING:

20.6 Arithmetic at Arbitrary Precision

Let’s compute the number \( \pi \) to a couple of thousand decimal places. In doing so, we’ll learn some things about multiple precision arithmetic on computers and meet quite an unusual application of the fast Fourier transform (FFT). We’ll also develop a set of routines that you can use for other calculations at any desired level of arithmetic precision.

To start with, we need an analytic algorithm for \( \pi \). Useful algorithms are quadratically convergent, i.e., they double the number of significant digits at each iteration. Quadratically convergent algorithms for \( \pi \) are based on the AGM (arithmetic geometric mean) method, which also finds application to the calculation of elliptic integrals (cf. §6.11) and in advanced implementations of the ADI method for elliptic partial differential equations (§19.5). Borwein and Borwein [1] treat this subject, which is beyond our scope here. One of their algorithms for \( \pi \) starts with the initializations

\[
X_0 = \sqrt{2} \\
\pi_0 = 2 + \sqrt{2} \\
Y_0 = \sqrt{2}
\]  

(20.6.1)