Quantum Mechanics,
Assignment - 2008

Part 1. Relativistic equations

I. Scalar particles, Klein-Gordon equation

- Write down the Klein Gordon equation for the radial function $\phi(r)$ of a particle with mass $m$ and charge $e$ in the Coulomb potential, i.e. presuming that

$$eA^\mu = (eA^0, 0, 0, 0), \quad eA^0 = -\frac{Z\alpha}{r} \quad (1.1)$$

Here $\alpha = e^2$ is the fine structure constant, $Z$ the charge of the centre, $\varepsilon$ is the energy of the particle in the Coulomb centre. $l, m$ are the momentum and its projection, $Y_{lm}(\theta, \phi)$ are spherical harmonics. Deriving equation for the radial function remember that the Laplacian when applied to the spherical harmonics $Y_{lm}(\theta, \phi)$ gives

$$\Delta\left[\phi(r)Y_{lm}(\theta, \phi)\right]=\left[\Delta,\phi(r)\frac{r}{l(l+1)}\phi(r)\right]Y_{lm}(\theta, \phi) \quad (1.2)$$

$$\Delta,\phi(r) = \phi^\prime(r) + \frac{2}{r}\phi^\prime(r)$$

- Verify that the equation for the radial wave function has the following form

$$E\phi = -\frac{1}{2m}\Delta,\phi + \frac{L(L+1)}{r^2}\phi - \frac{\zeta\alpha}{r}\phi \quad (1.3)$$

where $E, L, \zeta$ are some parameters, which can be expressed in terms of the energy $\varepsilon$, orbital momentum $l$ and charge $Z$. The form of Eq. (1.3), which look similar to the Schrödinger equation, inspires one to think of $E, L, \zeta$ as some “effective” energy, “effective” orbital momentum, and “effective” charge.

- Express $E, L, \zeta$ as functions of $\varepsilon, l, Z$.

Hint: it is convenient to use notation

$$\gamma = \sqrt{(l+1/2)^2 - (Z\alpha)^2} \quad (1.4)$$

Since the differential operator in Eq.(1.3) is similar to the one in the conventional Schrödinger equation for the Coulomb problem, its eigenvalue $E$ can be written using the Rydberg formulae

$$E = -\frac{Z^2\alpha^2}{(n_r+1+L)^2}, \quad n_r = 0,1,\ldots \quad (1.5)$$

Here $n_r$ is the radial quantum number.

- Use Eq. (1.5) to prove that the energies of the Coulomb problem for the Klein-Gordon equation satisfy the Sommerfeld formulae
\[ \varepsilon = \frac{mc^2}{\left[1 + Z^2\alpha^2/(n + \gamma - l - 1/2)^2\right]^{1/2}}, \quad n = 1, 2, \ldots \]  

(1.6)

Here \( n \) is the main quantum number.

- Prove that in the non-relativistic region (small \( Z \)) the Sommerfeld formulae recovers the Rydberg result. Find the first relativistic correction to the Rydberg result.
- Find the largest possible \( Z \), for which the Coulomb problem for the Klein-Gordon equation is well formulated.

II. Fermions, Dirac equation.

- Write down the Dirac equation for the free electron
- Verify that the second-order differential equation, which follows from the Dirac equation is identical to the Klein-Gordon equation.
- Verify that in the standard representation of the Dirac matrixes, when
  \[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \]  
  the following condition
  \[ \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \]  
  is satisfied.
- Write down the Dirac equation for the electron in an external electromagnetic field.
- Show that the later equation can be presented as the second-order differential equation
  \[ \left(p_\mu - eA_\mu\right)\left(p^\mu - eA^\mu\right)\psi - \frac{ie}{2} F_{\mu\nu} \sigma^{\mu\nu} \psi = m^2 \psi \]  
  where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) and
  \[ \sigma^{\mu\nu} = \frac{1}{2} \left( \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right) \]  
  \[ \frac{1}{2} F_{\mu\nu} \sigma^{\mu\nu} = \alpha \cdot E + i \Sigma \cdot B \]  
  where in the standard representation the Dirac matrixes read
  \[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} \]  
  \[ (2.6) \]

Part 2. Scattering theory

Consider propagation of a (nonrelativistic) particle in the s-wave (\( l = 0 \)) in an attractive potential

\[ U(r) = \begin{cases} -U < 0, & r < a \\ 0, & r > a \end{cases} \]  

(1.1)
• Solve the radial Schrödinger equation for the s-wave, presuming that

\[ R_{k,0} = \begin{cases} \frac{2}{r} \sin(Kr + \delta_0), & r > a \\ \text{const}, & r = 0 \end{cases} \]  

(1.2)

• Find an explicit equation, which defines the scattering phase \( \delta_0 \).

• Find conditions on the potential, which lead to a resonance at low energies. Remember that the low-energy resonance manifests itself through the following behaviour of the phase

\[ \cot \delta_0 = -\frac{\kappa}{K}, \quad |\kappa|, \ K \ll \frac{1}{a} \]  

(1.3)