Circular motion

Angular position $\theta$  how far it has turned

$eg$: it has turned through $90^\circ$

$\theta = 90^\circ, \theta = \pi/2$ radians

Angular velocity $\omega$  how fast it is turning

$eg$: it is turning at 3000 revolutions per minute

$\omega = 3000 \text{ rpm} = \frac{3000 \text{ turns}}{60 \text{ s}}$

$= 50 \text{ turns/s} = \frac{50 \times 2\pi}{s} = 314 \text{ rad.s}^{-1}$

Angular acceleration $\alpha$  how fast $\omega$ is increasing

$eg$: it goes from 0 to 3000 r.p.m in 5 seconds

$\omega_{\text{init}} = 0, \omega_{\text{final}} = 314 \text{ rad.s}^{-1}$

average $\alpha \equiv \frac{\omega_{\text{final}} - \omega_{\text{init}}}{\Delta t}$

Uniform circular motion

Circular motion with $\omega = \text{const}$. Even if $\alpha = 0$, This actually produces acceleration. eg bus going round a corner

Or consider hammer thrower

Resultant force produces acceleration in the horizontal direction, towards the centre of the motion
Centripetal force, centripital acceleration
Uniform circular motion

\[
\begin{align*}
\text{As } \Delta t \text{ and } \Delta \theta \to 0, \Delta x \to \text{ right angles to } \mathbf{y} \\
\therefore \mathbf{a} &= \lim_{\Delta t \to 0} \left( \frac{\Delta \mathbf{y}}{\Delta t} \right) \quad (// \ - \mathbf{r} \quad \text{centripital acceleration})
\end{align*}
\]

As \( \Delta \theta \to 0, \Delta s \to r \Delta \theta \)
\[
\begin{align*}
v &= \frac{ds}{dt} \\
&= r \frac{d\theta}{dt} = r \omega
\end{align*}
\]
\(\omega\) is the **angular velocity**
\[
|\Delta \mathbf{y}| = |v \Delta \theta| \quad (\text{n.b.: } |\Delta \mathbf{y}| \neq |\Delta \mathbf{x}|)
\]
\[
\lim_{\Delta t \to 0} |\Delta \mathbf{v}| = |\mathbf{v} \Delta \theta|
\]
\[
|\mathbf{a}| = \frac{|\Delta \mathbf{v}|}{\Delta t} = \mathbf{v} \frac{d\theta}{dt} = \mathbf{v} \omega
\]
\[
a = \frac{v^2}{r} = \omega^2 r \quad \text{but } \mathbf{a} // -\mathbf{r}
\]
so \( \mathbf{a} = -\omega^2 \mathbf{r} \)

**Circular motion:**
\[
\begin{align*}
\text{If } \theta \text{ measured in radians,} \\
s &= r \theta. \\
\therefore \quad v &= \frac{ds}{dt} = r \frac{d\theta}{dt} = r \omega \\
v &= r \omega \quad \omega = \frac{v}{r}
\end{align*}
\]
\[
\begin{align*}
a &= \frac{dv}{dt} = r \frac{d\omega}{dt} = r \alpha \\
a &= r \alpha \quad \alpha = \frac{a}{r}
\end{align*}
\]
### Motion with constant \( \alpha \).  

**Analogies**  

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\[
\begin{align*}
  v_f &= v_i + at \\
  \omega_f &= \omega_i + \alpha t \\
  \Delta x &= v_i t + \frac{1}{2} at^2 \\
  \omega_i^2 &= \omega_i^2 + 2\alpha \Delta \theta \\
  \Delta x &= \frac{1}{2} (v_i + v_f) t \\
  \Delta \theta &= \frac{1}{2} (\omega_i + \omega_f) t
\end{align*}
\]

**Example.** Centrifuge, initially spinning at 5000 rpm, slows uniformly to rest over 30 s. (i) What is its angular acceleration? (ii) How far does it turn while slowing down? (iii) How far does it turn during the first second of deceleration?

#### i) \( \omega_f = \omega_i + \alpha t \)

\[
= -17.5 \text{ rad.s}^{-2}.
\]

#### ii) \( \Delta \theta = \frac{1}{2} (\omega_i + \omega_f) t \)

\[
= 1,250 \text{ revolutions}
\]

#### iii) \( \Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2 \)

\[
= 82 \text{ turns}
\]

**Example** A bicycle wheel has \( r = 40 \text{ cm} \). What is its angular velocity when the bicycle travels at 40 km.hr\(^{-1}\)?

\[
\begin{align*}
  v &= \frac{ds}{dt} \\
  &= \frac{rd\theta}{dt} \\
  &= r\omega \\
  \omega &= \frac{v}{r}
\end{align*}
\]
**Torque**

Force applied at point displaced from axis of rotation.

(Note: if $F$ were only force $\Rightarrow$ acceleration:

How does the 'turning tendency' depend on $F$, $r$, $\theta$?

To get $\alpha$ but $a = 0$, need $\sum F = 0$.

- $F$ does not contribute to the turning about axis.

Consider rotation about $z$ axis

Only the component $F \sin \theta$ tends to turn

$$\tau = r (F \sin \theta)$$

or

$$\tau = F (r \sin \theta) = F r_\perp$$

where $r_\perp$ is called the moment arm

**Example** What is the maximum torque I apply by standing on a wheel brace 300 mm long?

$$\tau = r (F \sin \theta)$$

$$\max \tau = r F$$

**Example:** bicycle and rider ($m = 80$ kg) accelerate at $2 \text{ ms}^{-1}$. Wheel with $r = 40$ cm. What is torque at wheel? Front sprocket has 50 teeth, rear has 25, what is torque applied by legs?

$$F_{\text{front}} = F_{\text{back}}$$

$$\frac{r_{\text{front}}}{r_{\text{back}}} = \frac{50}{25}$$
Moment of inertia  
(Rotational analogue of mass)

Choose frame so that axis of rotation is at origin

\[
\mathbf{K} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots
\]
\[
= \frac{1}{2} m_1 (r_1 \omega_1)^2 + \frac{1}{2} m_2 (r_2 \omega_2)^2 + \ldots
\]
\[
= \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2 \quad (cf \ K = \frac{1}{2} m v^2)
\]

Define the **Moment of inertia**
(also called moment of inertia)

System of masses \( I = \sum m_i r_i^2 \)

Continuous body \( I = \int_{\text{body}} r^2 \, dm \)

I depends on total mass, distribution of mass, shape and **axis of rotation**. Units are kg.m^2

**Example** What is I for a hoop about its axis?

\[
\begin{array}{c}
M \\
\end{array}
\]

All the mass is at radius r, so

\[ I = Mr^2 \]

For a disc: \[ I = \int_{\text{body}} r^2 \, dm = \ldots = \frac{1}{2} MR^2 \]

For a sphere \[ I = nMR^2 \]  
\[ = M \left( \sqrt{n} R \right)^2 = Mk^2 \] where \( k = \sqrt{n} R \)

\[ I \equiv Mk^2 \] defines the **radius of gyration** \( k \)

\( k \) is the radius of a hoop with the same I as the object in question

<table>
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<tr>
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<th>I</th>
<th>k</th>
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<tr>
<td>hoop</td>
<td>MR^2</td>
<td>R</td>
</tr>
<tr>
<td>disc</td>
<td>( \frac{1}{2} MR^2 )</td>
<td>( \frac{R}{\sqrt{2}} )</td>
</tr>
<tr>
<td>solid sphere</td>
<td>( \frac{2}{5} MR^2 )</td>
<td>( \sqrt{\frac{2}{5}} R )</td>
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Example Use a flywheel to store the K of a bus at stops. Disc \( R = 80 \text{ cm}, \ M = 1 \text{ tonne.} \) How fast must it turn to store all the kinetic energy of a 10 t. bus at 60 km.hr\(^{-1}\)?

\[
\begin{align*}
v_m & = 60 \text{ km.hr}^{-1} \\
\omega_m & = 0 \\
v_s & = 0 \quad \text{not rolling} \\
\omega_s & = \ ? \text{ rev.s}^{-1}
\end{align*}
\]

Newton’s law for rotation

\[
\tau_{\text{total}} = I \alpha \quad \text{cf} \quad F_{\text{total}} = ma
\]

Example. What constant torque would be required to stop the earth’s rotation in one revolution? (Assume earth uniform.)

Plan: Know \( M, \ R, \ \omega_i, \ \omega_f, \ \Delta \theta \). Need \( \tau \).

Use \( \tau = I \alpha \), where \( \omega_i, \ \omega_f, \ \Delta \theta \rightarrow \alpha \)

Example Space-walking cosmonaut (\( m = 80 \text{ kg}, \ k = 0.3 \text{ m} \) about short axes) throws a 2 kg ball (from shoulder) at 31 ms\(^{-1}\) (\( \mathbf{x} \) displaced 40 cm from c.m.). How fast does she turn? Is this a record?

In orbit so no ext torques so \( \mathbf{L} \) conserved

\[
L_i = L_f = L_{\text{ball}} + L_{\cos}
\]

Analogies: linear and rotational kinematics

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kinematic equations

\[
\begin{align*}
v_f & = v_i + at \\
\omega_f & = \omega_i + \alpha t \\
\Delta x & = v_i t + \frac{1}{2} at^2 \quad \Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2 \\
v_f^2 & = v_i^2 + 2a\Delta x \quad \omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta \\
\Delta x & = \frac{1}{2} (v_i + v_f)t \quad \Delta \theta = \frac{1}{2} (\omega_i + \omega_f)t
\end{align*}
\]
Analogies: linear and rotational mechanics

mass \( m \) rotational inertia \( I \)

\[ I = \Sigma m r_i^2 \quad I = \int r^2 dm \]

Work & energy

\[ W = \int F \cdot ds \quad W = \int \tau \cdot d\theta \]

\[ K = \frac{1}{2} M v^2 \quad K = \frac{1}{2} I \omega^2 \]

force \( F \) torque \( \tau = rF \sin \theta \)

momentum \( \mathbf{p} = m \mathbf{v} \) angular momentum \( |\mathbf{L}| = mr v \sin \theta \)

Newton 2:

\[ \mathbf{F} = \frac{d}{dt} \mathbf{p} = m \mathbf{a} \quad \mathbf{\tau} = \frac{d}{dt} \mathbf{L} = I \mathbf{\alpha} \]

if \( m \) const if \( I \) const

Conservation of \( \mathbf{p} \) and \( \mathbf{L} \):

If no external forces torques act on a system, its momentum angular momentum is conserved.

Conservation of mechanical energy: if non-conservative forces and torques do no work, mechanical energy is conserved.

---

Example. Object mass \( m \) suspended by two strings as shown. Find \( T_1 \) and \( T_2 \).

It's not accelerating vertically so

\[ N_2 \rightarrow \Sigma F_y = ma_y = 0 \]

\[ \therefore \quad T_1 + T_2 - mg = 0 \quad (i) \]

It's not accelerating horizontally so

\[ N_2 \rightarrow \Sigma F_x = ma_x = 0 \]

\[ \therefore \quad 0 = 0 \quad not\ enough\ equations \]

It's not rotationally accelerating so:

\[ N_2 \rightarrow \Sigma \tau = I \alpha = 0 \]

\( \tau \) about c.m. clockwise

\[ \therefore \quad \tau_1 + \tau_2 = T_2 D - T_1 d = 0 \]

\[ T_1 + \frac{d}{D} T_1 - mg = 0 \rightarrow \quad T_1 = \frac{mg}{1 + \frac{d}{D}} \quad T_2 = \frac{mg}{1 + \frac{D}{d}} \]
Example Conical pendulum (Uniform circular motion.) What is the frequency?

Apply Newton 2 in two directions:
Vertical: \( a_y = 0 \) \( \Rightarrow \Sigma F_y = 0 \)

\[ T \cos \theta - W = 0 \]

\[ T = \frac{mg}{\cos \theta} \]

Horizontal:

\[ \frac{mv^2}{r} = a_c = T \sin \theta \]

\[ = \frac{mg \sin \theta}{\cos \theta} \]

\[ \Rightarrow \frac{v^2}{r} = g \tan \theta \]

\[ \Rightarrow v = \sqrt{rg \tan \theta} \]

\[ \Rightarrow \frac{2\pi}{T} = \sqrt{rg \tan \theta} \]

\[ \Rightarrow f = \frac{1}{2\pi} \sqrt{\frac{g \tan \theta}{r}} \]

Example Where is centre of earth-moon orbit?

\( F_e = F_m = F_g \) equal and opposite

each makes a circle about common centre of mass

\( F_g = m_m a_m = m_m \omega^2 r_m \)

\( F_g = m_e a_e = m_e \omega^2 r_e \)

\[ \Rightarrow \frac{r_e}{r_m} = \frac{m_m}{m_e} = \frac{5.98 \times 10^{24} \text{ kg}}{7.36 \times 10^{22} \text{ kg}} = 81.3 \]

"earth-moon distance" \( r_e + r_m = 3.85 \times 10^8 \text{ m} \)

solve \( r_m = 3.80 \times 10^8 \text{ m}, \ r_e = 4.7 \times 10^6 \text{ m} \)

\( \Rightarrow \) centre of orbit is inside earth
(Weight) = − (the force exerted by scales)

At poles, \( F - N = 0 \)

At latitude \( \theta \), \( F - N = ma \)

where \( a = r\omega^2 = (R_e \cos \theta)\omega^2 \)

\[ = \cdots = 0.03 \text{ ms}^{-1} \text{ at equator} \]

\[ = 0 \text{ at poles} \]

We define \( g = \frac{N}{m} = \frac{F - ma}{m} \)

**Question.** How can one buy and sell gold at different latitudes so as to make a profit?

**Example:** In what orbit does a satellite remain above the same point on the equator?

*Called the Clarke Geosynchronous Orbit*

i) Period of orbit = period of earth's rotation

\[ T = 23.9 \text{ hours} \]

\[ T^2 = \left( \frac{4\pi^2}{GM} \right) r^3 \]

\[ r = \sqrt[3]{\frac{GMT^2}{4\pi^2}} = \cdots \]

\[ = 44,000 \text{ km} \text{ popular orbit!} \]