Mechanics and forces

Aristotle: \( v = 0 \) is "natural" state
Galileo & Newton: \( a = 0 \) is "natural" state

Newton's Laws

First  "zero (total) force \( \Rightarrow \) zero acceleration"

more formally:
If \( \Sigma F = 0 \), \( \exists \) reference frames in which \( a = 0 \)
called **Inertial frames**

In such frames:

Second  \( \Sigma F = \frac{d}{dt} \mathbf{p} \)

\[
\frac{d}{dt} \mathbf{p} = \frac{d}{dt} m \mathbf{v} = \frac{d}{dt} m \mathbf{v} + m \frac{dv}{dt}
\]

if \( m \) is constant  \( \Sigma F = ma \)

(\( \Sigma F_x = m a_x \)  \( \Sigma F_y = m a_y \)  \( \Sigma F_z = m a_z \))

Third:  "To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal and directed to contrary parts"

Or

Forces always occur in pairs, \( \mathbf{F} \) and \( -\mathbf{F} \), one acting on each of a pair of interacting bodies.

\[
\mathbf{F}_{AB} \quad \mathbf{m}_A \quad \mathbf{m}_B \quad \mathbf{F}_{BA}
\]

Third  \( \mathbf{F}_{AB} = -\mathbf{F}_{BA} \)

Why so?

\[
\mathbf{F}_{AB} \quad \mathbf{m}_A \quad \mathbf{m}_B \quad \mathbf{F}_{BA}
\]

Work and energy

Work defined as \( dW = \mathbf{F}.ds \)

Work energy theorem follows from (is another way of stating) the 2nd law:

If \( m = \) constant

\[
\mathbf{F} = \frac{d}{dt} \mathbf{p} = \frac{d}{dt} m \mathbf{v} \quad m \frac{d}{dt} \mathbf{v}
\]

\( \therefore \) work done = \( \int_i^f \mathbf{F}.ds = \int_i^f m \frac{dv}{dt}.ds \)

\( = m \int_i^f ds \cdot \frac{dv}{dt} = m \int_i^f \mathbf{v}.d\mathbf{v} \)

Total work done = \( \frac{1}{2} mv_i^2 - \frac{1}{2} mv_f^2 \)

If work is done by a conservative force, define the potential energy \( U \)

Work done against conservative force = \( \Delta U \)

\( \therefore \) If only conservative forces act  \( U_i + \frac{1}{2} mv_i^2 = U_f + \frac{1}{2} mv_f^2 \)

Important forces:

\( \mathbf{F}_{grav} = -G \frac{Mm}{r^2} \hat{r} \)

On surface of the earth:  \( \mathbf{F}_{grav} = -G \frac{M_{earth}m}{r_{earth}^2} \hat{u}p \)

\( = m\left(\frac{GM_{earth}}{r_{earth}^2}\right) \) down  \( \equiv mg \) down
where \( g \equiv \left( \frac{GM_{\text{Earth}}}{r_{\text{Earth}}^2} \right) = \ldots = 9.8 \text{ ms}^{-2} \)

**Electric force** \( q_1 q_2 \rightarrow \hat{r} \)

\[
F_{\text{elec}} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}
\]

**van der Waals force** (electrodynamic force)

\[
F_{\text{vdw}} \propto \frac{1}{r^6}
\]

always attractive

**Phases of matter**

\[
\text{KE} = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 = \frac{3}{2}kT
\]

High T \( kT >> |\text{PE}| \) \( \rightarrow \) gas

Intermediate T \( kT \sim |\text{PE}| \) \( \rightarrow \) liquid

Low T \( kT << |\text{PE}| \) \( \rightarrow \) solid

Very High T \( kT > \text{work function} \) \( \rightarrow \) plasma

\( kT > \text{nuclear energies} \) \( \rightarrow \) exotic matter

**Hooke's Law.**

\[
\begin{align*}
\int_{r_{\text{spring}}}^{r_{\text{applied}}} & \Rightarrow \quad \text{Under tension} \\
\int_{x=0}^x & \Rightarrow \quad \text{No applied force}
\end{align*}
\]

\( F_{\text{spring}} \) in opposite direction to \( x \).

Experimentally, \(|F_s| \propto |x|\) over small range of \( x \)

\( F = -kx \) \hspace{1cm} **Hooke's Law.**

**Properties of condensed phases**

Inter-atomic & intermolecular forces and energies

\[
F = \begin{cases} 
\text{repulsive} & \text{attractive} \\
\text{Hooke's law} & \text{parabolic minimum in } U(r)
\end{cases}
\]

\( \rightarrow \) **Linear elasticity** \( \text{ (parabolic minimum in } U(r)) \)

Linear approximation to inter-molecular forces

**Stress** \( \sigma \equiv F/A \)

**Strain** \( \varepsilon \equiv \text{dimensionless change, e.g. } \frac{\delta L}{L} \)
Hooke's Law: \[ \frac{\sigma}{\varepsilon} = \text{elastic modulus} \]

**Longitudinal stress:**

- \[ F \]
- \[ A \]
- \[ L \]
- \[ \delta L \]

\[ \text{Young's modulus} \quad Y = \frac{F/A}{\delta L/L} = \frac{FL}{A\delta L} \]

**Bulk modulus**

\[ \kappa = -\frac{P}{\delta V/V} \]

**Poisson's ratio**

\[ \nu = -\frac{\varepsilon_{\text{lat.}}}{\varepsilon_{\text{long.}}} = -\frac{\delta d/d}{\delta L/L} \]

**Rigidity modulus** (shear modulus)

\[ G = \frac{F/A}{\delta L/L} = \frac{\tau}{\gamma} \]

**Example**

\[ F \text{ applied to the ends. } P \text{ applied to other 4 sides.} \]
What is \( P \) so that cross section \( A \) is unchanged?

\[ \varepsilon_y = -\nu\sigma_y/Y + \sigma_y/Y - \nu\sigma_z/Y \]
\[ 0 = +\nu F/YA - P/Y + \nu P/Y \]
\[ \nu F/A = P(1 - \nu) \]
\[ P = \frac{F}{A}\frac{\nu}{(1 - \nu)} \]
Example

For the potential \( U(r) = -\frac{A}{r^n} + \frac{B}{r^m} \), what is \( F(r) \)?

What is the \( r_0 \) for mechanical equilibrium?

\[
\frac{dU}{dr} = -\frac{dW}{ds} \rightarrow
\]

\[
F = -\frac{dU}{dr}
\]

\[
F(r) = n \frac{A}{r^{n+1}} - m \frac{B}{r^{m+1}}
\]

\( F = 0 \) at minimum of energy (\( dU = 0 \))

\[
\frac{n}{r_0^{n+1}} = \frac{m}{r_0^{m+1}}
\]

\[
\frac{r_0^{m+1}}{r_0^{n+1}} = r_0^{m-n} = \frac{mB}{nA}
\]

\[
r_0 = \left( \frac{mB}{nA} \right)^{1/(m-n)}
\]

\[
U(r_0) = -\frac{A}{r_0^n} + \frac{B}{r_0^m} = \ldots = \text{binding energy}
\]

\( \text{cf max force} \)

Example Using the polynomial model for \( U(r) \), determine the Young’s modulus \((C&L \ p43)\)

\[\frac{dF}{dr} = \frac{F/A}{\delta x/x} \]

\[= \frac{F/y^2}{\delta x/r_0} \]

\[= \frac{dF r_0}{dr y^2} \]

As before \( U(r) = -\frac{A}{r^n} + \frac{B}{r^m} \)

\[
F(r) = n \frac{A}{r^{n+1}} - m \frac{B}{r^{m+1}}
\]

\[
r_0 = \left( \frac{mB}{nA} \right)^{1/(m-n)}
\]

\[
\frac{d}{dr} F(r) = \frac{d}{dr} \left( n \frac{A}{r^{n+1}} - m \frac{B}{r^{m+1}} \right)
\]

\[= -n(n+1) \frac{A}{r^{n+2}} + m(m+1) \frac{B}{r^{m+2}} \]

at \( r = r_0 \),

\[
\frac{d}{dr} F(r) = \ldots = \frac{nA}{r_0^{n+2}} (m - n)
\]

\[
Y = \frac{dF r_0}{dr y^2} = \frac{(m - n)nA}{r_0^{n+1} y^2}
\]

Example. A steel set screw (\( Y = 206 \) GPa) has 50 turns, diameter 3 mm, and is just long enough. From contact, it is tightened one complete turn. Estimate the axial force it exerts.

\[Y = \frac{\sigma}{\varepsilon} = \text{stress/strain}
\]

\( \varepsilon \) here is \( \delta L/L \equiv 1/50 \).

\[
\sigma = F/A = F/\pi r^2
\]

\[
F = \pi r^2 \sigma = \pi \left( \frac{d}{2} \right)^2 Y \varepsilon
\]

\[= \ldots = 29 \text{ kN} \]
Inter-atomic & intermolecular forces

**Ionic solids**: electrostatic force

**Hydrogen bonds**: H has +δ charge

**van der Waals attraction**: attraction between transient dipoles $\propto r^{-6}$ at short range

**Crystalline solids**

**Packing factor**: fraction of space occupied by touching hard spheres

**Example** Calculate packing factor and $\rho$ of FCC

Face diagonal = $4r$
Side of unit cube = $4r/\sin 45^\circ$
\[= 2\sqrt{2}r\]

\[
p.f. = \frac{\text{Volume of spheres in unit cube}}{\text{Volume of unit cube}}
= \frac{8 \text{ corners} + 6 \text{ faces}}{(2\sqrt{2}r)^3}
= \frac{(8 \times \frac{1}{8} + 6 \times \frac{1}{2}) \frac{4}{3}\pi r^3}{(2\sqrt{2}r)^3}
= 74%\]

\[
\rho = \frac{(8 \times \frac{1}{8} + 6 \times \frac{1}{2}) \text{ atomic mass}}{a^3}
= \frac{4m}{a^3}
\]

**Ionic crystals**

e.g. NaCl: ions of similar size, each ion has six neighbours of opposite charge (coordination number six)

(It's like two interlaced FCCs)

**Covalent crystals**: share outer electrons. Depends on the angles of the electron orbitals

**Metallic crystal**: 'ions' in a sea of shared electrons. Often close packed in FCC or HCP. This, gives high $\rho$,
especially if atomic number is large (e.g. Au, Pt)

Hexagonal close packing
(vertical axis expanded here)

Intermediate bond types:

covalent

ionic

metallic

Amorphous solids (a.k.a. glass, vitreous phase)

Metallic glass. Cool the metal very quickly, e.g. small drops in liquid N$_2$.

Polymers

Long chains repeating one unit e.g. poly(ethylene) PE
(what sort of hydrocarbon is this?)

poly(vinyl acetate) PVA

Long flexible chains: usually tangle rather than crystallise, especially if they have side groups. Attractive force is vDW (and tangling)

Amorphous (partly) Crystalline

Crystalline polymers are only partly crystallised. Need uniform cross section, e.g. poly(vinylidene chloride)

Cross linking. Chemical bonds rigidify 3D structure. e.g. resins, vulcanisation in rubber (S bonds).
**Point defect**
vacancy, interstice, substitution

**Line defects**  Edge and screw

**Plane defects, espec. microcracks**
Surface tension and surface energy

Molecule in bulk is uniformly attracted in each direction. Molecule at surface has ~ no attraction to atmosphere. :: work done against the nett force in order to make a surface.

Work to make new surface is done against surface tension $\gamma$.

\[ \gamma = \text{force per unit length in the plane of the surface} \]

\[ (\text{e.g. raise wire dy}) \]

\[ F = 2\gamma L \quad (\text{two sides}) \]

\[ \text{Work } \quad dW = F \cdot dy = 2\gamma L \cdot dy \]

\[ \frac{dA}{2Ldy} = 2Ldy \quad (\text{two sides}) \]

\[ \gamma = \frac{dW}{dA} \]

So surface tension = surface free energy per unit area

Contact angles and menisci

The surface energy of solid-vapour is $S_S$. Solid-liquid is $S_{LS}$

Surface tension of liquid is $\gamma$

Contact angle $\theta$

Suppose that the meniscus rises as shown. Work done by tension

\[ dW = -\gamma L dx \cos \theta = -\gamma dA \cos \theta \]

\[ = \text{new solid-liquid energy} - \text{solid-vapour energy} \]

\[ = S_{LS}dA - S_SdA \]

\[ \gamma \cos \theta = S_S - S_{LS} \]

Some surface free energies

<table>
<thead>
<tr>
<th>Class</th>
<th>Material</th>
<th>$S$ [J m$^{-2}$] \left( = \frac{\gamma}{N m^{-1}} \right)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid</td>
<td>water</td>
<td>0.073 \quad (0.073)</td>
</tr>
<tr>
<td></td>
<td>Hg</td>
<td>0.051 \quad (0.051)</td>
</tr>
<tr>
<td>Glass</td>
<td>SiO$_2$</td>
<td>4.4 \quad (4.4)</td>
</tr>
<tr>
<td>Ionic</td>
<td>NaCl</td>
<td>0.5 \quad (0.5)</td>
</tr>
<tr>
<td>solid</td>
<td>KCl</td>
<td>0.11 \quad (0.11)</td>
</tr>
<tr>
<td>Mica</td>
<td>in air</td>
<td>0.38 \quad (0.38)</td>
</tr>
<tr>
<td></td>
<td>in vacuum</td>
<td>5 \quad (5)</td>
</tr>
<tr>
<td>Covalent</td>
<td>Al$_2$O$_3$ (sapphire)</td>
<td>6-32 \quad (6-32)</td>
</tr>
<tr>
<td>solid</td>
<td>C (diamond)</td>
<td>5.24 \quad (5.24)</td>
</tr>
<tr>
<td>Metal</td>
<td>Zn</td>
<td>0.105 \quad (0.105)</td>
</tr>
<tr>
<td></td>
<td>Pb</td>
<td>0.763 \quad (0.763)</td>
</tr>
<tr>
<td></td>
<td>Si</td>
<td>1.24 \quad (1.24)</td>
</tr>
<tr>
<td>Polycrystalline</td>
<td>SiC</td>
<td>32 \quad (32)</td>
</tr>
<tr>
<td></td>
<td>Graphite</td>
<td>68 \quad (68)</td>
</tr>
<tr>
<td></td>
<td>Granite</td>
<td>200 \quad (200)</td>
</tr>
<tr>
<td></td>
<td>Cast Iron</td>
<td>1,520 \quad (1,520)</td>
</tr>
</tbody>
</table>
Pressure excess inside balanced by surface tension
\[ \Delta P \pi r^2 = \gamma 2\pi r \]
\[ \Delta P = \frac{\gamma}{r} \]

**Young-Laplace Equation**

**Surface energy of solids**

Complicated by mechanical strength of materials. Work to cleave
\[ = U_{\text{surf}} + U_{\text{bend}} \]
Put in expression for deflection of cantilever spring:
\[ F = 3Yd^3y^2 \]
\[ L = \frac{8x^4}{8x^4} \]

**Diverse comments about various types of materials and behaviour**

Be careful talking of 'strength of materials'

- Sometimes you want high E (\(\phi\) small \(e\))
- Other times you want high \(\sigma_{\text{max}}\)
- Yet other times you want low E (\(\phi\) large \(e\))

Macroscopic \(\sigma_{\text{max}}\) usually << microscopic \(\sigma_{\text{max}}\)

- Composite materials aim to minimise the difference by limiting propagation of dislocations
- In composites, \(\sigma_{\text{micro}} \neq \sigma_{\text{macro}}\)

**Ductility** refers to the ease of plastic deformation without rupture.

- Sometimes 'good', sometimes 'bad'

Work hardening: 'dislocation tangle'. Dislocation stops at slip plane,
- fewer moveable dislocations \(\phi\), less ductility

Ductile rupture: break bonds, eventually form macroscopic rupture
For some materials, mechanical properties may depend on time scale (rheology).

For some materials, mechanical properties depend on temperature. Plastic deformation is easier at high T, brittle fracture is more likely at low T.

Material properties may also be changed by:
- Chemical reactions
  - Radiation (UV or even light from some polymers, $\gamma$ and X for other materials)
  - Hydration (for hydrophilic, fibrous materials)