Rotation

PHYS1121-1131  UNSW 2010 Session 1.

Which wins: car or ball?
Why? (insert your answer)

Kinetic energy of a rotating body

Choose frame so that axis of rotation is at origin

\[ K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots \]
\[ = \frac{1}{2} m_1 (r_1 \omega_1)^2 + \frac{1}{2} m_2 (r_2 \omega_2)^2 + \ldots \]
\[ = \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2 \quad (cf \ K = \frac{1}{2} m v^2) \]

Rotational analogue of mass:

Define the **Moment of inertia**

System of masses \[ I = \sum m_i r_i^2 \]

Continuous body \[ I = \int_{\text{body}} r^2 \, dm \]

I depends on total mass, distribution of mass, shape and **axis of rotation**. Units are kg.m^2
Example  What is I for a hoop about its axis?

All the mass is at radius r, so  \( I = Mr^2 \)

For a disc: \( I = \int_{\text{body}} r^2 \, dm = \ldots = \frac{1}{2} MR^2 \)

For a sphere \( I = \ldots = \frac{2}{5} MR^2 \)

Note

\[ I = nMR^2 \quad \text{where } n \text{ is a number} \]

\[ = M \left( \sqrt{n} R \right)^2 = Mk^2 \quad \text{where } k = \sqrt{n} R \]

\( I = Mk^2 \) defines the **radius of gyration** \( k \)

\( k \) is the radius of a hoop with the same \( I \) as the object in question

<table>
<thead>
<tr>
<th>Object</th>
<th>( I )</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>hoop</td>
<td>( MR^2 )</td>
<td>( R )</td>
</tr>
<tr>
<td>disc</td>
<td>( \frac{1}{2} MR^2 )</td>
<td>( \frac{R}{\sqrt{2}} )</td>
</tr>
<tr>
<td>solid sphere</td>
<td>( \frac{2}{5} MR^2 )</td>
<td>( \sqrt{\frac{2}{5}} R )</td>
</tr>
</tbody>
</table>
**Example** Use a flywheel to store the K of a bus at stops. Disc R = 80 cm, M = 1 tonne. How fast must it turn to store all the kinetic energy of a 10 t. bus at 60 km.hr⁻¹?

Moving (subscript m), stopped (subscript s)

\[
v_m = 60 \text{ km.hr}^{-1} \quad \quad v_s = 0 \quad \text{not rolling}
\]
\[
\omega_m = 0 \quad \quad \omega_s = ? \text{ rev.s}^{-1}
\]

\[
K_m = K_s
\]
\[
\frac{1}{2} M_{bus} v_m^2 = \frac{1}{2} I_{disc} \omega_s^2
\]
\[
M_{bus} v_m^2 = \frac{1}{2} M_{disc} R^2 \omega_s^2
\]
\[
\omega_s = \frac{v_m}{R} \sqrt{\frac{2M_{bus}}{M_{disc}}}
\]
\[
= 90 \text{ rad.s}^{-1} = 900 \text{ rpm} \quad (\text{revolutions per minute})
\]
**Rolling vs skidding:**

---

**Example** A bicycle wheel has \( r = 40 \text{ cm} \). What is its angular velocity when the bicycle travels at 40 km.hr\(^{-1}\)?

\[
\omega = \frac{v}{r} = \frac{40000 \text{ m/3600 s}}{0.4 \text{ m}} = 28 \text{ rad.s}^{-1} \quad (= 4.4 \text{ turns/second})
\]

Axle travels at \( v \)

Point of contact stationary

Top of wheel travels 2\( v \) (see *Rolling on Physclips*)
Example. A solid sphere, a disc and a hoop roll down an inclined plane. Which travels fastest?

Rolling: point of application of friction stationary \( \therefore \) non-conservative forces do no work \( \therefore \).

\[
U_f + K_f = U_i + K_i
\]

\[
0 + \left( \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 \right) = Mgh + 0
\]

\[
\omega = \frac{v}{R} \quad \text{and write} \quad I = Mk^2
\]

\[
\frac{1}{2} Mv^2 + \frac{1}{2} Mk^2 \frac{v^2}{R^2} = Mgh
\]

\[
\frac{1}{2} v^2 \left( 1 + \frac{k^2}{R^2} \right) = gh
\]

\[
v = \sqrt{\frac{2gh}{1 + k^2/R^2}}
\]

\[
\frac{k_{\text{sphere}}}{R} = \sqrt{\frac{2}{5}} < \frac{k_{\text{disc}}}{R} = \sqrt{\frac{1}{2}} < \frac{k_{\text{hoop}}}{R} = 1
\]

\[
\therefore \quad v_{\text{sphere}} > v_{\text{disc}} > v_{\text{hoop}} \quad r \text{ doesn’t appear, so the result is independent of size}
\]
Rotational kinematics:

\[ r \text{ is constant} \]

If \( \theta \) measured in radians,

\[ s = r\theta. \hspace{2cm} \text{(definition of angle)} \]

\[ \therefore \hspace{0.5cm} v = \frac{ds}{dt} = r \frac{d\theta}{dt} = r\omega \]

\[ v = r\omega \hspace{2cm} \text{or} \hspace{0.5cm} \omega = \frac{v}{r} \]

\[ \therefore \hspace{0.5cm} a = \frac{dv}{dt} = r \frac{d\omega}{dt} = r\alpha \]

\[ a = r\alpha \hspace{2cm} \text{or} \hspace{0.5cm} \alpha = \frac{a}{r} \]

Motion with constant \( \alpha \).

<table>
<thead>
<tr>
<th>Analogies</th>
<th>Linear</th>
<th>Angular</th>
</tr>
</thead>
<tbody>
<tr>
<td>displacement</td>
<td>( x )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>velocity</td>
<td>( v )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>acceleration</td>
<td>( a )</td>
<td>( \alpha )</td>
</tr>
</tbody>
</table>

\[ v_f = v_i + at \hspace{2cm} \omega_f = \omega_i + \alpha t \]

\[ \Delta x = v_it + \frac{1}{2} at^2 \hspace{2cm} \Delta \theta = \omega_it + \frac{1}{2} \alpha t^2 \]

\[ v_f^2 = v_i^2 + 2a\Delta x \hspace{2cm} \omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta \]

\[ \Delta x = \frac{1}{2} (v_i + v_f) t \hspace{2cm} \Delta \theta = \frac{1}{2} (\omega_i + \omega_f) t \]

*Derivations identical - see previous. Need only remember one version.*
Example. Centrifuge, initially spinning at 5000 rpm, slows uniformly to rest over 30 s. (i) What is its angular acceleration? (ii) How far does it turn while slowing down? (iii) How far does it turn during the first second of deceleration? (rpm = revolutions per minute)

i) \[ \omega_f = \omega_i + \alpha t \]
\[ \alpha = \frac{\omega_f - \omega_i}{t} \]
\[ = \frac{0 - \frac{5000 \times 2\pi \text{ rad}}{60 \text{ s}}}{30 \text{ s}} \]
\[ = -17.5 \text{ rad.s}^{-2}. \]

ii) \[ \Delta \theta = \frac{1}{2} (\omega_i + \omega_f) t \]
\[ = \frac{1}{2} (0 + 5000 \text{ rpm}) \times 0.5 \text{ min} \]
\[ = 1.250 \text{ revolutions} \]

iii) \[ \Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2 \]
\[ = \frac{5000 \times 2\pi \text{ rad}}{60 \text{ s}} \times (1 \text{ s}) - \frac{1}{2} (17.5 \text{ rad.s}^{-2}) \times (1 \text{ s})^2 \]
\[ = 515 \text{ rad} \quad (= 82 \text{ turns}) \]
**What causes angular acceleration?**

Force applied at point displaced from axis of rotation.

![Force diagram](image)

(Note: if $F$ were only force $\Rightarrow$ acceleration:

How does the 'turning tendency' depend on $F$, $r$, $\theta$?

To get $\alpha$ but $a = 0$, need $\Sigma F = 0$.

$-F$ does not contribute to the turning about axis.

**Torque.** *(rotational analogue of force)*

Consider rotation about $z$ axis

![Torque diagram](image)

Only the component $F \sin \theta$ tends to turn

$$\tau = r (F \sin \theta) \quad (r \text{ component of } F)$$

or

$$= F (r \sin \theta) = F r_\perp \quad (F \text{ component of } r)$$

where $r_\perp$ is called the moment arm

**Example** What is the maximum torque I apply by standing on a wheel spanner 300 mm long?

$$\tau = r (F \sin \theta)$$

$$\max \tau = r F$$

$$= 0.3 \text{ m} \times 700 \text{ N} = 200 \text{ Nm}$$

if it still doesn't move: lift, use both hands or jump on it
The vector product.

\[ \mathbf{a} \times \mathbf{b} \equiv ab \sin \theta \]
\( \mathbf{a} \times \mathbf{b} \) at right angles to \( \mathbf{a} \) and \( \mathbf{b} \) in right hand sense

pronounced "a cross b"

For right hand

**Thumb \times index = middle**  
(remember TIM)  
(or North \times East = down  
remember NED)

Turn screwdriver from \( \mathbf{a} \) to \( \mathbf{b} \) and (r.h.) screw moves in direction of \( \mathbf{a} \times \mathbf{b} \)

Apply to unit vectors:

\[ |\mathbf{i} \times \mathbf{i}| = 1.1 \sin 0^\circ = 0 = \mathbf{i} \times \mathbf{i} = \mathbf{k} \times \mathbf{k} \]
\[ |\mathbf{i} \times \mathbf{j}| = 1.1 \sin 90^\circ = 1 = |\mathbf{j} \times \mathbf{k}| = |\mathbf{k} \times \mathbf{i}| \]
\[ \mathbf{i} \times \mathbf{j} = \mathbf{k} \]
\[ \mathbf{j} \times \mathbf{k} = \mathbf{i} \]
\[ \mathbf{k} \times \mathbf{i} = \mathbf{j} \]

but \( \mathbf{i} \times \mathbf{i} = -\mathbf{k} \]
\( \mathbf{k} \times \mathbf{j} = -\mathbf{i} \]
\( \mathbf{i} \times \mathbf{k} = -\mathbf{j} \)

Usually evaluate by \( |\mathbf{a} \times \mathbf{b}| = ab \sin \theta \)

but **Vector product by components** is neat

\[ \mathbf{a} \times \mathbf{b} = (ax \mathbf{i} + ay \mathbf{j} + az \mathbf{k}) \times (bx \mathbf{i} + by \mathbf{j} + bz \mathbf{k}) \]
\[ = (axbx) \mathbf{i} \times \mathbf{i} + (ayby) \mathbf{j} \times \mathbf{i} + (azbz) \mathbf{k} \times \mathbf{i} \]
\[ + (axby) \mathbf{i} \times \mathbf{j} + (aybz) \mathbf{j} \times \mathbf{k} + (azbx) \mathbf{k} \times \mathbf{i} \]
\[ + (aybx) \mathbf{j} \times \mathbf{i} + (azby) \mathbf{k} \times \mathbf{i} + (axby) \mathbf{i} \times \mathbf{k} \]

\[ \mathbf{a} \times \mathbf{b} = (axby - aybx) \mathbf{k} + (aybz - azby) \mathbf{i} + (azbx - axby) \mathbf{j} \]

Example.

\( \mathbf{F} = (3 \mathbf{i} + 5 \mathbf{j}) \text{N}, \quad \mathbf{r} = (4 \mathbf{j} + 6 \mathbf{k}) \text{m}; \quad \mathbf{\tau} = ? \)

\[ \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \]
\[ = (rxFy - ryFx) \mathbf{k} + (ryFz - rzFy) \mathbf{i} + (rzFx - rxFz) \mathbf{j} \]
\[ = (0 - 4 \text{ mN}) \mathbf{k} + (0 - 6 \text{ mN}) \mathbf{i} + (6 \text{ mN} - 0) \mathbf{j} = (-30 \mathbf{i} + 18 \mathbf{j} - 12 \mathbf{k}) \text{Nm} \]
Example: bicycle and rider ($m = 80$ kg) accelerate at $2$ m/s$^2$. Wheel with $r = 40$ cm. What is torque at wheel?

\[ F_{\text{ext}} = ma \]
\[ \tau = rF_{\text{ext}} \sin \theta = rF \]
\[ = rma = ... = 64 \text{ Nm. horizontal} \]

Front sprocket has 50 teeth, rear has 25, what is torque applied by legs?

\[ F_{\text{front}} = F_{\text{back}} \]
\[ \frac{r_{\text{front}}}{r_{\text{back}}} = \frac{50}{25} \]
\[ \frac{\tau_{\text{front}}}{\tau_{\text{back}}} = \frac{r_{\text{front}}F_{\text{front}}}{r_{\text{back}}F_{\text{back}}} = 2. \]
\[ \tau_{\text{front}} = 128 \text{ Nm} \text{ horizontal} \text{ why larger?} \]

Newton's law for rotation

System of particles, $m_i$, all rotating with same $\omega$ and $\alpha$ about same axis. $r_i$ is perpendicular distance from the axis of rotation.

\[ \tau_i = r_i \times F_i \]
\[ \tau_i = r_i F_{ti} \]

where $F_{ti}$ is the tangential component of $F$

\[ \tau_i = r_i m_i \dot{a}_i \]
\[ \tau_i = r_i m_i a_i \]
\[ = r_i m_i r_i a_i \]
\[ \sum \tau_i = \sum m_i r_i^2 \alpha_i \]

but all $\alpha_i = \alpha$

so \[ \tau_{\text{total}} = I \alpha \]

and \(\tau, \alpha\) on axis

Newton's law for rotation

\[ \tau_{\text{total}} = I \alpha \]

\(\text{compare with}\) \(F_{\text{total}} = ma\)
Example. What constant torque would be required to stop the earth's rotation in one revolution? (Assume earth uniform.)

Plan: Know $M$, $R$, $\omega_i$, $\omega_f$, $\Delta \theta$. Need $\tau$.

Use $\tau = I \alpha$, where $\omega_i$, $\omega_f$, $\Delta \theta \rightarrow \alpha$

$\omega_f = 0$, $\omega_i = \frac{2\pi}{23h56min} = 7.27 \times 10^{-5} \text{ rad.s}^{-1}$

$\omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta$ (cf $v_f^2 = v_i^2 + 2a\Delta x$)

$\alpha = \frac{\omega_f^2 - \omega_i^2}{2\Delta \theta}$

$\tau = I \alpha = \frac{2}{5} MR^2 \frac{\omega_f^2 - \omega_i^2}{2\Delta \theta}$

$= ...$

$= 4 \times 10^{28} \text{ Nm}$

Example Mass $m$ on string on drum radius $r$ on a wheel with radius of gyration $k$ and mass $M$. How long does it take to turn 10 turns? solve for $a$ or $\alpha$, use kinematic equations.

N2 for m (vertical): $mg - T = ma$

N2 for wheel:

$\tau = I \alpha$

$rT \sin 90^\circ = Mk^2 \frac{a}{r}$

$T = Ma \left(\frac{k}{r}\right)^2$

$mg - Ma \left(\frac{k}{r}\right)^2 = ma$

$a = \frac{mg}{m + M \left(\frac{k}{r}\right)^2}$

$\Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2$ (cf $\Delta x = v_i t + \frac{1}{2} at^2$)

$t = \sqrt{\frac{2\Delta \theta}{\alpha}} = \sqrt{\frac{2(20\pi \text{ rad}) \left(1 + \frac{M \left(\frac{k}{r}\right)^2}{m} \right)}{g} r}$
**Example.** Rod rotates about one end. Which reaches bottom first: m or the end of the rod?

![Diagram of rod and masses](image)

Acceleration of end of rod is

\[ a = L\alpha \]

For rod, \( \tau = I\alpha \)

so

\[ a = L \frac{\tau}{I} \]

For rod about an end, \( I = \frac{1}{3} ML^2 \).

Mg acts at c.m. so \( \tau = Mg \frac{L}{2} \)

\[ a = L \cdot \frac{Mg \frac{L}{2}}{\frac{1}{3} ML^2} = \frac{3}{2} g \]

*Why do falling chimneys break?*

**Example** A car is doing work at a rate of 20 kW and travelling at 100 kph. Wheels are \( r = 30 \) cm. What is the (total) torque applied by the drive wheels?

\[ P = Fv, \text{ so by analogy: } P = \tau \omega \]

Wheels are rolling so \( \omega = \frac{v}{r} \)

\[ \therefore \tau = \frac{P}{\omega} = \frac{Pr}{v} \]

\[ = \frac{2 \times 10^4 \text{ W \ 0.3 m}}{10^5 \text{ m/3600 s}} \]

\[ = 220 \text{ Nm} \]

*(not equal to torque on tail shaft or at flywheel)*

**Important note:** There is not a lot of rotational mechanics in our syllabus: we don’t have angular momentum. So the following material is not in the syllabus. I’m including it, however, because some of you will certainly come across it later. As you’ll see, there are lots of analogies with linear mechanics so, except for the vector product, it is not tricky.
Angular momentum

For a particle of mass \( m \) and momentum \( \mathbf{p} \) at position \( \mathbf{r} \) relative to origin \( O \) of an inertial reference frame, we define angular momentum (w.r.t.) \( O \)

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p}
\]

or

\[
\mathbf{L} = rp \sin \theta
\]

Example  What is the angular momentum of the moon about the earth?

\[
\begin{align*}
L &= |\mathbf{r} \times \mathbf{p}| \\
&= r p \sin \theta \\
&= r m v \sin 90^\circ \\
&= m r^2 \omega \\
&= (7.4 \times 10^{22} \text{ kg}) (3.8 \times 10^8 \text{ m})^2 \frac{2\pi}{27.3 \times 24 \times 3600 \text{ s}} \\
&= 2.8 \times 10^{34} \text{ kg m}^2 \text{ s}^{-1}
\end{align*}
\]

Direction is North

Example  Two trains mass \( m \) approach at same speed \( v \), travelling antiparallel, on tracks separated by distance \( d \). What is their total angular momentum, as a function of separation, about a point halfway between them?

\[\begin{align*}
\mathbf{L}_1 &= \mathbf{r}_1 \times \mathbf{p}_1 \\
&= \mathbf{r}_1 \times m \mathbf{v}_1 \\
|\mathbf{L}_1| &= (d/2)m v \text{ (clockwise on my diagram)}
\end{align*}\]

\[\begin{align*}
|\mathbf{L}_2| &= (d/2)m v \text{ (also clockwise)} \\
&= d m v \text{ independent of separation}
\end{align*}\]
Newton 2 for angular momentum:

\[ \Sigma \tau = r \times F = r \times \frac{d}{dt} p \]

\[ \frac{d}{dt} \mathbf{L} = \frac{d}{dt} (r \times p) \]

\[ = \left( \frac{d}{dt} r \right) \times p + r \times \frac{d}{dt} p \]

Remember: Order important in vector multiplication!

\[ = \mathbf{v} \times m\mathbf{v} + \tau \]

\[ \Sigma \tau = \frac{d}{dt} \mathbf{L} \]

Newton 2 in rotation

Question: A top balances on a sharp point. Why doesn't it fall over? (Qualitative treatment only.)

![Diagram of a top balancing on a sharp point](image)

\[ \tau = \frac{d}{dt} \mathbf{L} \]

\[ \frac{d}{dt} \mathbf{L} \parallel \tau \]

but \( \tau \) is horizontal

so \( d\mathbf{L} \) is perpendicular to \( g \)

Also boomerangs, frisbees, satellites

Systems of particles

Total angular momentum \( \mathbf{L} \)

\[ \mathbf{L} = \Sigma (r_i \times p_i) \]

\[ \frac{d}{dt} \mathbf{L} = \Sigma \frac{d}{dt} (r_i \times p_i) \]

\[ = \Sigma \tau_i \]

\[ = \Sigma \tau_i \text{ internal} + \Sigma \tau_i \text{ external} \]

Internal torques cancel in pairs (Newton 3)

\[ \therefore \Sigma \tau_{\text{ext}} = \frac{d}{dt} \mathbf{L} \]

\( \tau_{\text{ext}} \) is the sum of all external torques.

(This equation derived for inertial frames but it is also true for other frames if centre of mass is taken as origin.)

Consequence:

If \( \Sigma \tau_{\text{ext}} = 0 \), \( \frac{d}{dt} \mathbf{L} = 0 \).

Conservation of angular momentum of isolated system
Example  Circular motion of ball on string. What happens to the speed of the ball as the string is shortened? (Neglect air resistance).

![Diagram of circular motion](image)

Tension *does* do work, but it doesn't exert torque \( (\tau // \mathbf{r}) \cdot \therefore \) angular momentum conserved.

\[
\mathbf{L}_i = \mathbf{r} \times \mathbf{p} = rp \sin \theta = rmv \sin \theta
\]

Example: Person on rotating seat holds two 2.2 kg masses at arms' length. Draws masses in to chest. What is \( \Delta \omega \)? Is \( K \) conserved?

Rough estimates: \( k_{\text{person}} \) about long axis \( \sim 15 \) cm

\[
\begin{align*}
I_p &= Mk^2 = \sim 70 \text{ kg}. (.15 \text{ m})^2 \quad \text{include moving part of chair} \\
I_p &\sim 1.6 \text{ kgm}^2 \\
I_m &= mr^2 = 2.2 \text{ kg}. (0.8 \text{ m})^2 \\
I_m &\approx 1.4 \text{ kgm}^2 \quad \text{(arms extended)} \\
I_m' &= mr^2 = 2.2 \text{ kg}. (0.2 \text{ m})^2 \\
&\approx 0.1 \text{ kgm}^2 \quad \text{(arms in)}
\end{align*}
\]

No external torques \( \Rightarrow L_i = L_f \)

\[
\begin{align*}
(I_p + 2I_m)\omega_i &= (I_p + 2I'_m)\omega_f \\
\frac{\omega_f}{\omega_i} &= \frac{I_p + 2I_m}{I_p + 2I'_m} \sim 2.4 \\
K_f &= \frac{1}{2} (I_p + 2I_m)\omega_f^2 \\
K_i &= \frac{1}{2} (I_p + 2I'_m)\omega_i^2 = 2.4
\end{align*}
\]

Arms do work: \( Fds = ma_{\text{centrip}}.ds \)
Example  Space-walking cosmonaut (m = 80 kg, k = 0.3 m about short axes) throws a 2 kg ball (from shoulder) at 31 m\(s^{-1}\) (\(\mathbf{v}\) displaced 40 cm from c.m.). How fast does she turn? Is this a record?

\begin{align*}
\text{In orbit so no ext torques so } L_c \text{ conserved} \\
L_i = L_f = L_{\text{ball}} + L_{\cos} \\
0 = \mathbf{r} \times m\mathbf{v} - I\omega \\
= rmv - Mk^2\omega \\
\omega = \frac{rmv}{Mk^2} \\
= 3.4 \text{ rad.s}^{-1} \\
= 33\frac{1}{3} \text{ r.p.m.}
\end{align*}

(Yes, it must be a record)

Questions

Can a docking spacecraft rotate without using rockets?

Can a cat, initially with L = 0, rotate while falling so as to land on its feet?
Summary  Analogies: linear and rotational kinematics

<table>
<thead>
<tr>
<th>Linear</th>
<th>Angular</th>
</tr>
</thead>
<tbody>
<tr>
<td>displacement</td>
<td>angular displacement</td>
</tr>
<tr>
<td>velocity</td>
<td>angular velocity</td>
</tr>
<tr>
<td>acceleration</td>
<td>angular acceleration</td>
</tr>
</tbody>
</table>

**kinematic equations**

\[
\begin{align*}
\nu_f &= \nu_i + at \\
\Delta \nu &= \nu_i t + \frac{1}{2} at^2 \\
\nu_f^2 &= \nu_i^2 + 2a\Delta \nu \\
\Delta \nu &= \frac{1}{2} (\nu_i + \nu_f) t \\
\omega_f &= \omega_i + \alpha t \\
\Delta \omega &= \omega_i t + \frac{1}{2} \alpha t^2 \\
\omega_f^2 &= \omega_i^2 + 2\alpha \Delta \omega \\
\Delta \omega &= \frac{1}{2} (\omega_i + \omega_f) t
\end{align*}
\]

**Analogies: linear and rotational mechanics**

mass  \( m \) rotational inertia  \( I \)

\[ I = \Sigma m_i r_i^2 \quad I = \int r^2 dm \]

Work & energy

\[
\begin{align*}
W &= \int F \cdot ds \\
W &= \int \tau \cdot d\theta \\
K &= \frac{1}{2} M\nu^2 \\
K &= \frac{1}{2} I\omega^2
\end{align*}
\]

force  \( \mathbf{F} \) torque  \( \mathbf{\tau} = \mathbf{r} \times \mathbf{F} \)

momentum  \( \mathbf{p} = m \mathbf{v} \) angular momentum  \( \mathbf{L} = m \mathbf{r} \times \mathbf{v} \)

Newton 2:

\[
\begin{align*}
\mathbf{F} &= \frac{d}{dt} \mathbf{p} = ma \\
\mathbf{\tau} &= \frac{d}{dt} \mathbf{L} = I\alpha
\end{align*}
\]

if \( m \) const  if \( I \) const

**Momentum**  \( \mathbf{p} = mv \)

Newton 1&2  \( \mathbf{F}_{\text{ext}} = \frac{d}{dt} \mathbf{p} \)

Conservation law:

If no external forces act  momentum conserved

If \( m \) constant,  \( \mathbf{F}_{\text{ext}} = \frac{d}{dt} \mathbf{p} = \frac{d}{dt} (mv) = ma \)

\[
\begin{align*}
\mathbf{F}_{\text{ext}} (r \sin \theta) &= \frac{d}{dt} (m \mathbf{r} \sin \theta)
\end{align*}
\]
Angular momentum \( L = (r \sin \theta) \, mv \)

Newton for rotation \( \tau_{\text{ext}} = \frac{d}{dt} \, L \)

Conservation law:

If no external torques act \( \text{angular momentum} \) conserved

If I constant, \( \tau_{\text{ext}} = \frac{d}{dt} \, L = \frac{d}{dt} \, I \omega = I \alpha \)

Conservation of \( p \) and \( L \):

If no external \( \text{forces} \) act on a system,

its \( \text{angular momentum} \) is conserved.

Conservation of mechanical energy: if non-conservative forces and torques do no work, mechanical energy is conserved

**Example** Particle mass \( m \) moves with \( r = (At) \hat{i} + B \hat{j} + \left( Ct - \frac{1}{2} gt^2 \right) \hat{k} \)

(i) What is \( p \) for the mass? (ii) What is its \( L \) about the origin? (iii) what torque \( \tau \) acts on it? (iv) What is the shape of this motion?

i) \( p = m \, v = m \frac{d}{dt} \, r \)
   \[ = m \left( A \, \hat{i} + (C - gt) \, \hat{k} \right) \]

ii) \( L = r \times p \)
    \[ \text{recall:} \]
    \[ r_x \quad r_y \quad r_z \quad r_x \]
    \[ p_x \quad p_y \quad p_z \quad p_x \]
    \[ \hat{i} \quad \hat{j} \quad \hat{k} \quad \hat{i} \]
    \[ = (r_x p_y - r_y p_x) \hat{k} + (r_y p_z - r_z p_y) \hat{i} + (r_z p_x - r_x p_z) \hat{j} \]
    \[ L = -B m A \hat{k} + B m (C-gt) \hat{j} + \]
    \[ \left( (Ct - \frac{1}{2} gt^2) m A - A m (C-gt) \right) \hat{i} \]
    \[ = B (C-gt) m \hat{i} + \frac{1}{2} A m g t^2 \hat{j} - A B m \hat{k} \]

(iii) \( \tau = \frac{d}{dt} \, L = -B m g \hat{i} + A m g t \hat{j} \)
Torsional pendulum.

Useful way of comparing unknown $I$ with that of a simple object (e.g. rod).
Object with $I$ is suspended on wire. The wire, when twisted, produces a restoring torque $\tau = -\kappa \theta$

$$\tau = I \alpha = I \frac{d^2 \theta}{dt^2}$$

$$\frac{d^2 \theta}{dt^2} = \frac{\tau}{I} = -\frac{\kappa}{I} \theta$$

solution is:

$$\theta = \theta_m \sin (\Omega t + \phi) \quad \text{where} \quad \Omega = \sqrt{\frac{\kappa}{I}}$$

Period

$$T = \frac{2\pi}{\Omega} = 2\pi \sqrt{\frac{I}{\kappa}}$$

where $\kappa$ is the const of the wire

.: for two different objects, $\frac{T_2^2}{T_1^2} = \frac{I_1}{I_2}$

---

Simple pendulum.

Mass $m$, suspended on light string. Radius of mass $r \ll M$ :. treat as particle.

N2 in vertical: $mg = T \cos \theta$

N2 in horizontal: $T \sin \theta = ma = -m \frac{d^2 x}{dt^2}$

If $\theta \ll 1$, $\sin \theta \approx \theta \approx \frac{x}{L}$, $\cos \theta \approx 1$.

$$m \frac{d^2 x}{dt^2} = -T \sin \theta = -mg \frac{x}{L}$$

$$\frac{d^2 x}{dt^2} = -T \sin \theta = -\frac{g}{L} x$$

solution is:

$$x = x_m \sin (\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{g}{L}}$$

Period

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}}$$
Physical pendulum.

Object, mass m, rotational inertia I, free to rotate.

N2 for rotation: \( \tau = I \alpha \)

\[-mg h \sin \theta = I \frac{d^2 \theta}{dt^2} \]

If \( \theta << 1 \),

\[ \frac{d^2 \theta}{dt^2} = -\frac{mgh}{I} \sin \theta \]

\[ \frac{d^2 \theta}{dt^2} \approx -\frac{mgh}{I} \theta \]

solution is:

\[ \theta = \theta_m \sin (\omega t + \phi) \quad \text{where} \quad \omega = \sqrt{\frac{mgh}{I}} \]

Period

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mgh}} \]

(put all mass at c.m. \( I = mk^2 = mh^2 \Rightarrow \) previous result)
Example.
Disc, mass m, radius R, suspended at point h from centre. What is T for this pendulum?

Period
\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{mg}} \]

Parallel axis theorem:
\[ I_{\text{new}} = I_{\text{cm}} + mh^2 \]
\[ = \frac{1}{2} mR^2 + mh^2 \]
\[ T = 2\pi \sqrt{\frac{\frac{1}{2} R^2 + h^2}{gh}} \]

(if h >> R, get \[ 2\pi \sqrt{\frac{h}{g}} \] as for simple pendulum
if h = 0, T → \( \infty \))

Example. Object mass m suspended by two strings as shown. Find \( T_1 \) and \( T_2 \).

It's not accelerating vertically so
N2 \( \rightarrow \) \[ \Sigma F_y = m \dot{y} = 0 \]
\[ \therefore \quad T_1 + T_2 - mg = 0 \quad (i) \]

It's not accelerating horizontally so
N2 \( \rightarrow \) \[ \Sigma F_x = m \dot{x} = 0 \]
\[ \therefore \quad 0 = 0 \quad \text{not enough equations} \]

It's not rotationally accelerating so:
N2 \( \rightarrow \) \[ \Sigma \tau = I \ddot{\alpha} = 0 \]
\[ \tau \text{ about c.m. clockwise } \quad \therefore \quad T_1 + \frac{d}{D} T_1 - mg = 0 \]
\[ T_1 = \frac{mg}{1 + d/D} \quad T_2 = \frac{mg}{1 + D/d} \]

i) A cyclist travels round a corner with a radius of 20 m, travelling at 30 kilometers per hour, on a horizontal road surface. Showing your working, determine the angle at which he should and the bicycle lean towards the centre of the turn, so as not to fall over. (The cyclist does not change his angle with respect to the bicycle as he rounds the corner, he is always symmetrically positioned with respect to the plane of symmetry of the bicycle.)

ii) If the coefficients of kinetic and static friction between the tyres and the road are 0.8 and 1.0 respectively,
what is the maximum speed at which the cyclist can take this corner?