Rotational kinematics. Physics UNSW. Joe Wolfe

If $\theta$ measured in radians,
\[
    s = r\theta. \quad \text{(definition of angle)}
\]
\[
    \therefore \quad v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega
\]
\[
    v = r\omega \quad \omega = \frac{v}{r}
\]
\[
    \therefore \quad a = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha
\]
\[
    a = r\alpha \quad \alpha = \frac{a}{r}
\]

**Motion with constant $\alpha$.**

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<td>Velocity</td>
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<td>Acceleration</td>
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\[
    v_f = v_i + at \quad \omega_f = \omega_i + \alpha t
\]
\[
    \Delta x = v_i t + \frac{1}{2} at^2 \quad \Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2
\]
\[
    v_i^2 = v_f^2 + 2a\Delta x \quad \omega_i^2 = \omega_f^2 + 2\alpha \Delta \theta
\]
\[
    \Delta x = \frac{1}{2} (v_i + v_f)t \quad \Delta \theta = \frac{1}{2} (\omega_i + \omega_f)t
\]

*Derivations identical - see previous. Need only remember one version*

**Example.** Centrifuge, initially spinning at 5000 rpm, slows uniformly to rest over 30 s. (i) What is its angular acceleration? (ii) How far does it turn while slowing down? (iii) How far does it turn during the first second of deceleration?

i) $\omega_f = \omega_i + \alpha t$ \quad \text{(cf $v_f = v_i + at$)}

\[
    \alpha = \frac{\omega_f - \omega_i}{t}
\]
\[
    = \frac{0 - \frac{5000\times 2\pi \text{ rad}}{60\text{s}}}{30\text{s}}
\]
\[
    = -17.5 \text{ rad.s}^{-2}.
\]

ii) $\Delta \theta = \frac{1}{2} (\omega_i + \omega_f)t$ \quad \text{(cf $\Delta x = \frac{1}{2} (v_i + v_f)t$)}

\[
    = \frac{1}{2} (0 + 5000\text{rpm}) \times 0.5 \text{ min}
\]
\[
    = 1,250 \text{ revolutions}
\]

iii) $\Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2$ \quad \text{(cf $\Delta x = v_i t + \frac{1}{2} at^2$)}

\[
    = \frac{5000 \times 2\pi \text{ rad}}{60\text{s}} \times (1\text{s}) - \frac{1}{2} (17.5 \text{ rad.s}^{-2})(1\text{s})^2
\]
\[
    = 515 \text{ rad} \quad (=82 \text{ turns})
\]
**Example**  A bicycle wheel has $r = 40$ cm. What is its angular velocity when the bicycle travels at $40$ km.hr$^{-1}$?

$$v = \frac{ds}{dt} = r \frac{d\theta}{dt}$$

**Important result**

$$\omega = \frac{v}{r} = \frac{40000 \text{ m/3600 s}}{0.4 \text{ m}} = 28 \text{ rad.s}^{-1} = 4.4 \text{ turns/second}$$

**What causes angular acceleration?**

Force applied at point displaced from axis of rotation.

(Note: if $F$ were only force $\Rightarrow$ acceleration:

How does the 'turning tendency' depend on $F, r, \theta$?

**Torque.**

* (rotational analogue of force)

Consider rotation about $z$ axis

Only the component $F \sin \theta$ tends to turn

$$\tau = r (F \sin \theta) \quad (r \ast \text{component of } F)$$

or

$$\tau = F (r \sin \theta) = F r_\perp \quad (F \ast \text{component of } r)$$

where $r_\perp$ is called the moment arm

**Example** What is the maximum torque I apply by standing on a wheel spanner 300 mm long?

$$\tau = r (F \sin \theta)$$

$$\max \tau = r F$$

$$\tau = 0.3 \text{ m} \ast 700 \text{ N} = 200 \text{ Nm}$$

*if it still doesn't move: lift, use both hands or jump on it*
Example: bicycle and rider \((m = 80 \text{ kg})\) accelerate at \(2 \text{ ms}^{-2}\). Wheel with \(r = 40 \text{ cm}\). What is torque at wheel?

\[
\begin{align*}
F_{\text{ext}} &= ma \\
\tau &= rF_{\text{ext}} \sin \theta = rF \\
&= rma = \ldots = 64 \text{ Nm.} \text{ horizontal}
\end{align*}
\]

Front sprocket has 50 teeth, rear has 25, what is torque applied by legs?

\[
\begin{align*}
F_{\text{front}} &= F_{\text{back}} \\
\frac{r_{\text{front}}}{r_{\text{back}}} &= \frac{50}{25} \\
\frac{\tau_{\text{front}}}{\tau_{\text{back}}} &= \frac{r_{\text{front}}F_{\text{front}}}{r_{\text{back}}F_{\text{back}}} = 2. \\
\tau_{\text{front}} &= 128 \text{ Nm} \text{ horizontal} \quad \text{why larger?}
\end{align*}
\]
Choose frame so that axis of rotation is at origin

\[
K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \ldots
\]

\[
= \frac{1}{2} m_1 (r_1 \omega_1)^2 + \frac{1}{2} m_2 (r_2 \omega_2)^2 + \ldots
\]

\[
= \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2 \quad (cf \ K = \frac{1}{2} mv^2)
\]

Define the Moment of Inertia

System of masses \[ I = \sum m_i r_i^2 \]

Continuous body \[ I = \int r^2 \, dm \]

I depends on total mass, distribution of mass, shape and axis of rotation. Units are kg.m²

Example What is I for a hoop about its axis?

All the mass is at radius r, so

\[ I = Mr^2 \]

For a disc: \[ I = \int_{body} r^2 \, dm = \ldots = \frac{1}{2} MR^2 \]

For a sphere \[ I = \frac{2}{5} MR^2 \]

\[ I = nMR^2 \quad n \text{ is a number} \]

\[ = M \left( \sqrt[n]{R} \right)^2 = Mk^2 \quad \text{where } k = \sqrt[n]{R} \]

\[ I = Mk^2 \quad \text{defines the radius of gyration } k \]

k is the radius of a hoop with the same I as the object in question

<table>
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<tr>
<th>object</th>
<th>I</th>
<th>k</th>
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<tr>
<td>hoop</td>
<td>MR²</td>
<td>R</td>
</tr>
<tr>
<td>disc</td>
<td>( \frac{1}{2} MR^2 )</td>
<td>( \frac{R}{\sqrt{2}} )</td>
</tr>
<tr>
<td>solid sphere</td>
<td>( \frac{2}{5} MR^2 )</td>
<td>( \sqrt{\frac{2}{5}} R )</td>
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Example Use a flywheel to store the K of a bus at stops. Disc R = 80 cm, M = 1 tonne. How fast must it turn to store all the kinetic energy of a 10 t. bus at 60 km.hr\(^{-1}\)?

\[
\begin{align*}
v_m &= 60 \text{ km.hr}^{-1} \\
\omega_m &= 0 \\
K_m &= K_s \\
\frac{1}{2} M_{bus} v_m^2 &= \frac{1}{2} I_{disc} \omega_s^2 \\
M_{bus} v_m^2 &= \frac{1}{2} M_{disc} R^2 \omega_s^2 \\
\omega_s &= \frac{v_m}{R} \sqrt{\frac{2 M_{bus}}{M_{disc}}} \\
&= 21 \text{ rad.s}^{-1} = 3.3 \text{ rev.s}^{-1} = 200 \text{ rpm}
\end{align*}
\]

Example. A solid sphere, a disc and a hoop roll down an inclined plane. Which travels fastest?

Rolling: point of application of friction stationary \(\therefore\) non-conservative forces do no work \(\therefore\):

\[
\begin{align*}
U_f + K_f &= U_i + K_i \\
0 + \left( \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2 \right) &= Mgh + 0 \\
\omega &= \frac{v}{R} \quad \text{and write} \quad I = Mk^2 \\
\frac{1}{2} M v^2 + \frac{1}{2} Mk^2 \frac{v^2}{R^2} &= Mgh \\
\frac{1}{2} v^2 \left( 1 + \frac{k^2}{R^2} \right) &= gh \\
v &= \sqrt{\frac{2gh}{1 + k^2/R^2}} \\
k_{\text{sphere}} = \sqrt{\frac{2}{5}} < k_{\text{disc}} = \sqrt{\frac{1}{2}} < k_{\text{hoop}} = 1
\end{align*}
\]

\(\therefore\) \(v_{\text{sphere}} > v_{\text{disc}} > v_{\text{hoop}} \quad \text{independent of size}\)
Newton's law for rotation
System of particles, \( m_i \), all rotating with same \( \omega \) and \( \alpha \) about same axis. \( r_i \) is perpendicular distance from the axis of rotation.

\[ \mathbf{F}_i = m_i \mathbf{a}_i \]

\[ \mathbf{r}_i = \mathbf{r}_i \times \mathbf{F}_i \]

where \( \mathbf{F}_i \) is the tangential component of \( \mathbf{F} \)

\[ \tau_i = r_i F_{ti} \]

\[ \tau_i = r_i m_i \alpha_i \]

\[ \tau_i = \sum m_i r_i^2 \alpha_i \] but all \( \alpha_i = \alpha \)

so \[ \tau_{\text{total}} = \frac{I \alpha}{\omega} \] and \( \tau, \alpha \) on axis

Example. What constant torque would be required to stop the earth's rotation in one revolution? (Assume earth uniform.)

**Plan:** Know \( M, R, \omega_i, \omega_f, \Delta \theta \). Need \( \tau \).

Use \( \tau = I \alpha \) where \( \omega_i, \omega_f, \Delta \theta \rightarrow \alpha \)

\[ \omega_f = 0, \quad \omega_i = \frac{2 \pi}{23h56min} = 1.72 \times 10^{-4} \text{ rad.s}^{-1} \]

\[ \omega_f^2 = \omega_i^2 + 2\alpha \Delta \theta \] (cf \( v_f^2 = v_i^2 + 2\alpha \Delta \theta \))

\[ \alpha = \frac{\omega_f^2 - \omega_i^2}{2 \Delta \theta} \]

\[ \tau = I \alpha = \frac{2}{5} MR^2 \frac{\omega_f^2 - \omega_i^2}{2 \Delta \theta} \]

\[ \tau = 4.1 \times 10^{28} \text{ Nm} \]
Example  Mass $m$ on string on drum radius $r$ on a wheel with radius of gyration $k$ and mass $M$. How long does it take to turn 10 turns?

Plan: solve for $a$ or $\alpha$, use kinematic equations.

N2 for $m$ (vertical): \[ mg - T = ma \]
N2 for wheel: \[ \tau = I\alpha \]
\[ rT = Mk^2 \frac{a}{r} \]
\[ T = Ma \left( \frac{k}{r} \right)^2 \]
\[ mg - Ma \left( \frac{k}{r} \right)^2 = ma \]
\[ a = \frac{mg}{m + M \left( \frac{k}{r} \right)^2} \]
$\Delta \theta = \omega t + \frac{1}{2} \alpha t^2$ \hspace{1cm} cf $\Delta x = v_0 t + \frac{1}{2} at^2$
\[ t = \sqrt{\frac{2\Delta \theta}{\alpha}} = \sqrt{\frac{2(20\pi \text{ rad}) \left( 1 + \frac{M}{m} \left( \frac{k}{r} \right)^2 \right) r}{g}} \]

Example  A car is doing work at a rate of 20 kW and travelling at 100 kph. Wheels are $r = 30$ cm. What is the (total) torque applied by the drive wheels?

$P = Fv$, so by analogy: \[ P = \tau \omega \]

Wheels are rolling so \[ \omega = \frac{v}{r} \]

\[ \therefore \tau = \frac{P}{\omega} = \frac{Pr}{v} \]
\[ = \frac{2 \times 10^4 \text{ W} \times 0.3 \text{ m}}{10^5 \text{ m/3600 s}} \]
\[ = 220 \text{ Nm} \]

(not equal to torque on tail shaft or at flywheel)
### Analogies: linear and rotational kinematics

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<td>angular displacement</td>
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<tr>
<td>velocity</td>
<td>angular velocity</td>
</tr>
<tr>
<td>acceleration</td>
<td>angular acceleration</td>
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#### Kinematic Equations

- **Linear**
  - Initial Velocity: \( v_i \)
  - Final Velocity: \( v_f \)
  - Acceleration: \( a \)
  - Time: \( t \)

- **Angular**
  - Initial Angular Velocity: \( \omega_i \)
  - Final Angular Velocity: \( \omega_f \)
  - Angular Acceleration: \( \alpha \)
  - Time: \( t \)

#### Examples:

- **Linear**
  - \( v_f = v_i + at \)
  - \( \Delta x = v_i t + \frac{1}{2}at^2 \)
  - \( v_f^2 = v_i^2 + 2at \)

- **Angular**
  - \( \omega_f = \omega_i + \alpha t \)
  - \( \Delta \theta = \omega_i t + \frac{1}{2} \alpha t^2 \)
  - \( \omega_f^2 = \omega_i^2 + 2 \alpha \Delta \theta \)

- **Linear**
  - \( \Delta x = \frac{1}{2} (v_i + v_f) t \)

- **Angular**
  - \( \Delta \theta = \frac{1}{2} (\omega_i + \omega_f) t \)

### Analogies: linear and rotational mechanics

- **Mass** \( m \)
- **Moment of Inertia** \( I \)

#### Work & Energy

- **Linear**
  - \( W = \int F \cdot ds \)
  - \( K = \frac{1}{2} M v^2 \)

- **Angular**
  - **Work**
    - \( W = \int \tau \cdot d\theta \)
  - **Kinetic Energy**
    - \( K = \frac{1}{2} I \omega^2 \)

- **Linear Momentum**
  - \( \mathbf{p} = m \mathbf{v} \)

- **Angular Momentum**
  - \( \mathbf{L} = \mathbf{r} \times \mathbf{p} \)
  - \( \mathbf{L} = m \mathbf{r} \times \mathbf{v} \)

#### Newton's Second Law:

- **Linear**
  - \( \mathbf{F} = \frac{d}{dt} \mathbf{p} = m \mathbf{a} \)
  - \( \mathbf{F} = \frac{d}{dt} \mathbf{L} = I \alpha \)
  - if \( m \) is constant

- **Angular**
  - \( \mathbf{T} = \frac{d}{dt} \mathbf{L} = I \alpha \)
  - if \( I \) is constant
Momentum \[ p = mv \]

Newton 1&2 \[ F_{\text{ext}} = \frac{d}{dt} p \]

Conservation law:
If no external forces act \[ \text{momentum conserved} \]
If \( m \) constant, \[ F_{\text{ext}} = \frac{d}{dt} p = \frac{d}{dt} (mv) = ma \]

Angular momentum \[ L \equiv (r \sin \theta) mv \]

Newton for rotation \[ \tau_{\text{ext}} = \frac{d}{dt} L \]

Conservation law:
If no external torques act \[ \text{angular momentum conserved} \]
If \( I \) constant, \[ \tau_{\text{ext}} = \frac{d}{dt} L = \frac{d}{dt} I\omega = I\alpha \]

Conservation of \( p \) and \( L \):
If no external forces or torques act on a system, its momentum and angular momentum is conserved.

Conservation of mechanical energy: if non-conservative forces and torques do no work, mechanical energy is conserved.
Example  Particle mass \( m \) moves with

\[
\mathbf{r} = (At) \mathbf{i} + B \mathbf{j} + \left( Ct - \frac{1}{2} gt^2 \right) \mathbf{k}
\]

(i) What is \( \mathbf{p} \) for the mass?  (ii)  What is its \( \mathbf{L} \) about the origin?  (iii)  what torque \( \mathbf{T} \) acts on it?  (iv)  What is the shape of this motion?

i) \[
\mathbf{p} = m \mathbf{v} = m \frac{d\mathbf{r}}{dt} = m (A \mathbf{i} + (C - gt) \mathbf{k})
\]

ii) \[
\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad \text{recall:}
\]

\[
\begin{array}{cccc}
\mathbf{r} & \mathbf{r}_x & \mathbf{r}_y & \mathbf{r}_z \\
\mathbf{p} & \mathbf{p}_x & \mathbf{p}_y & \mathbf{p}_z \\
\mathbf{k} & \mathbf{i} & \mathbf{j} & \mathbf{k} \\
\mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i}
\end{array}
\]

\[
= (r_x p_y - r_y p_x) \mathbf{k} + (r_y p_z - r_z p_y) \mathbf{i} + (r_z p_x - r_x p_z) \mathbf{j}
\]

\[
\mathbf{L} = -BmA \mathbf{k} + Bm(C-gt)\mathbf{i} + \left( Ct - \frac{1}{2} gt^2 \right) mA - Atm(C-gt)\mathbf{k}
\]

= \( B(C-gt)m \mathbf{i} + \frac{1}{2} Amgt^2 \mathbf{j} - ABm \mathbf{k} \)

(iii) \[
\mathbf{T} = \frac{d}{dt} \mathbf{L} = -Bmg \mathbf{i} + Amgt \mathbf{j}
\]

\[\text{Torsional pendulum.}\]

Useful way of comparing unknown \( I \) with that of a simple object (e.g. rod).

Object with \( I \) is suspended on wire. The wire, when twisted, produces a restoring torque \( \tau = -\kappa \theta \)

\[
\tau = I \alpha = I \frac{d^2 \theta}{dt^2}
\]

\[
\frac{d^2 \theta}{dt^2} = \frac{\tau}{I} = -\frac{\kappa}{I} \theta
\]

solution is:

\[
\theta = \theta_m \sin (\Omega t + \phi) \quad \text{where} \quad \Omega = \sqrt{\frac{\kappa}{I}}
\]

Period

\[
T = \frac{2\pi}{\Omega} = 2\pi \sqrt{\frac{I}{\kappa}} \quad \text{where} \ \kappa \ \text{is the const of the wire}
\]

\[
\therefore \ \text{for two different objects,} \quad \frac{T_2^2}{T_1^2} = \frac{I_1}{I_2}
\]
Example. String round drum \((r)\) on spool \((R)\). What is the critical angle \(\theta\) which determines direction of motion?

a) If it slides (kinetic friction \(F_k\)) it moves right.
b) If it rolls (static friction \(F_s\)) it moves left.

N2 vertical: \[ T \sin \theta + N = W \] (i)

N2 horizontal: \[ T \cos \theta = F_{fr} \] (ii)

N2 rot\(\text{n}\) about centre \[ T_r = F_{fr}R \] (iii)

At point of sliding, \[ F_{fr} = \mu_s N \] (iv)

Unknowns: \(T, \theta, F_{fr}, N\). Substitute (iv):

(ii) \[ T \cos \theta = \mu_s N \]

(iii) \[ T_r = \mu_s NR \]

\[ \cos \theta = \frac{r}{R} \]

---

Simple pendulum.

*You may meet this in the lab....*

Mass \(m\), suspended on light string. Radius of mass \(r \ll M\): treat as particle.

N2 in vertical: \[ mg = T \cos \theta \]

N2 in horizontal: \[ T \sin \theta = ma = -m \frac{d^2x}{dt^2} \]

If \(\theta << 1\), \(\sin \theta \approx \theta \approx \frac{x}{L}, \cos \theta \approx 1\).

\[ m \frac{d^2x}{dt^2} = -T \sin \theta = -mg \frac{x}{L} \]

\[ \frac{d^2x}{dt^2} = -T \sin \theta = -\frac{g}{L} x \]

Solution is:

\[ x = x_m \sin (\omega t + \phi) \text{ where } \omega = \sqrt{\frac{g}{L}} \]

Period

\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \]
Physical pendulum.
Object, mass m, rotational inertia I, free to rotate.

N2 for rotation: \[ \tau = I \alpha \]
\[-mg \, h \sin \theta = I \frac{d^2 \theta}{dt^2} \]

If \( \theta \ll 1 \),
\[ \frac{d^2 \theta}{dt^2} = -\frac{mg}{I} \sin \theta \]
\[ \frac{d^2 \theta}{dt^2} \approx -\frac{mg}{I} \theta \]

solution is:
\[ \theta = \theta_m \sin (\omega t + \phi) \text{ where } \omega = \sqrt{\frac{mg}{I}} \]

Period
\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{1}{mgh}} \]

(put all mass at c.m. \( I = mk^2 = mh^2 \Rightarrow previous \ result)\]

Example.
Disc, mass m, radius R, suspended at point h from centre. What is T for this pendulum?

Period
\[ T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{1}{mgh}} \]

Parallel axis theorem:
\[ I_{\text{new}} = I_{\text{cm}} + mh^2 \]
\[ = \frac{1}{2} mR^2 + mh^2 \]
\[ T = 2\pi \sqrt{\frac{\frac{1}{2} R^2 + h^2}{gh}} \]

(if \( h \gg R \), get \( 2\pi \sqrt{\frac{h}{g}} \) as for simple pendulum \( \text{if } h = 0, T \to \infty ) \)
Example. Object mass $m$ suspended by two strings as shown. Find $T_1$ and $T_2$.

It's not accelerating vertically so
\[ N_2 \rightarrow \sum F_y = m\ddot{y} = 0 \]
\[ \therefore T_1 + T_2 - mg = 0 \quad (i) \]

It's not accelerating horizontally so
\[ N_2 \rightarrow \sum F_x = m\ddot{x} = 0 \]
\[ \therefore 0 = 0 \quad \text{not enough equations} \]

It's not rotationally accelerating so:
\[ N_2 \rightarrow \sum \tau = I\alpha = 0 \]
\[ \tau \text{ about c.m. clockwise} \quad \therefore \tau_1 + \tau_2 = T_2D - T_1d = 0 \]
\[ T_1 + \frac{d}{D}T_1 - mg = 0 \]
\[ T_1 = \frac{mg}{1 + d/D} \quad T_2 = \frac{mg}{1 + D/d} \]
The vector product (not in syllabus).

\[ \mathbf{a} \times \mathbf{b} \]

Define \( |\mathbf{a} \times \mathbf{b}| = ab \sin \theta \)

\( \mathbf{a} \times \mathbf{b} \) at right angles to \( \mathbf{a} \) and \( \mathbf{b} \) in right hand sense

pronounced "a cross b"

For right hand

Thumb \times\ \text{index} = \text{middle} \hspace{1cm} \text{(remember TIM)}

(or \ \text{North} \times\ \text{East} = \text{down} \ \text{remember NED})

Turn screwdriver from \( \mathbf{a} \) to \( \mathbf{b} \) and (r.h.) screw moves in direction of \( (\mathbf{a} \times \mathbf{b}) \)

Apply to unit vectors:

\[ |\mathbf{i} \times \mathbf{j}| = 1.1 \sin 0^\circ = 0 = \mathbf{j} \times \mathbf{i} = \mathbf{k} \times \mathbf{i} \]

\[ |\mathbf{i} \times \mathbf{j}| = 1.1 \sin 90^\circ = 1 = |\mathbf{j} \times \mathbf{k}| = |\mathbf{k} \times \mathbf{i}| \]

\[ \mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i} \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \]

but \[ \mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j} \]

Usually evaluate by \( |\mathbf{a} \times \mathbf{b}| = ab \sin \theta \)

but \ \textbf{Vector product by components} \ is neat

\[ \mathbf{a} \times \mathbf{b} = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \times (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \]

\[ = (a_x b_x) \mathbf{i} \times \mathbf{i} + (a_y b_y) \mathbf{j} \times \mathbf{j} + (a_z b_z) \mathbf{k} \times \mathbf{k} \]

\[ + (a_x b_y) \mathbf{i} \times \mathbf{j} + (a_y b_z) \mathbf{j} \times \mathbf{k} + (a_z b_x) \mathbf{k} \times \mathbf{i} \]

\[ + (a_y b_x) \mathbf{j} \times \mathbf{i} + (a_z b_y) \mathbf{k} \times \mathbf{j} + (a_x b_z) \mathbf{i} \times \mathbf{k} \]

\[ \mathbf{a} \times \mathbf{b} = (a_x b_y - a_y b_x) \mathbf{k} + (a_y b_z - a_z b_y) \mathbf{i} + (a_z b_x - a_x b_z) \mathbf{j} \]

Example.

\[ \mathbf{F} = (3 \mathbf{i} + 5 \mathbf{j}) \text{N}, \quad \mathbf{r} = (4 \mathbf{j} + 6 \mathbf{k}) \text{m}; \quad \mathbf{L} = ? \]

\[ \mathbf{L} = \mathbf{r} \times \mathbf{F} \]

\[ = (r_x F_y - r_y F_x) \mathbf{k} + (r_y F_z - r_z F_y) \mathbf{i} + (r_z F_x - r_x F_z) \mathbf{j} \]

\[ = (0 - 4 \text{ m. 3 N}) \mathbf{k} + (0 - 6 \text{ m. 5 N}) \mathbf{i} + (6 \text{ m. 3 N} - 0) \mathbf{j} \]

\[ = (-30 \mathbf{i} + 18 \mathbf{j} - 12 \mathbf{k}) \text{Nm} \]

Angular momentum

For a particle of mass \( m \) and momentum \( \mathbf{p} \) at position \( \mathbf{r} \) relative to origin \( O \) of an inertial reference frame, we define angular momentum (w.r.t.) \( O \)

\[ \mathbf{L} = \mathbf{r} \times \mathbf{p} \]
Example  What is the angular momentum of the moon about the earth?
\[
L = \left| \mathbf{r} \times \mathbf{p} \right| \\
= rp \sin \theta \\
\approx rmv \sin 90^\circ \\
= mr^2 \omega
\]
\[
= (7.4 \times 10^{22} \text{ kg}) (3.8 \times 10^8 \text{ m})^2 \frac{2\pi}{27.3 \times 24 \times 3600 \text{ s}}
\]
\[
= 2.8 \times 10^{34} \text{ kg m}^2 \text{ s}^{-1}
\]
**Direction** is North

Example  Two trains mass m approach at same speed v, travelling antiparallel, on tracks separated by distance d. What is their total angular momentum, as a function of separation, about a point halfway between them?
\[
\mathbf{L}_1 = \mathbf{n} \times \mathbf{p}_1 \\
= \mathbf{n} \times m\mathbf{v}_1 \\
= (d/2)m\mathbf{v} \quad \text{(clockwise on my diagram)}
\]
\[
\mathbf{L}_2 = (d/2)m\mathbf{v} \quad \text{(also clockwise)}
\]
\[
= dm\mathbf{v} \quad \text{independent of separation}
\]

Newton 2 for angular momentum:
\[
\sum \mathbf{\tau} \equiv \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d}{dt} \mathbf{p}
\]
\[
\frac{d}{dt} \mathbf{L} = \frac{d}{dt} (\mathbf{r} \times \mathbf{p})
\]
\[
= \left( \frac{d}{dt} \mathbf{r} \right) \times \mathbf{p} + \mathbf{r} \times \frac{d}{dt} \mathbf{p} \quad \text{order important!}
\]
\[
= \mathbf{v} \times m\mathbf{v} + \mathbf{r}
\]
\[
\sum \mathbf{\tau} = \frac{d}{dt} \mathbf{L} \quad \text{Newton 2 in rotation}
\]
Question: A top balances on a sharp point. Why doesn't it fall over?  
(Qualitative treatment only.)

\[ \mathbf{T} = \frac{d}{dt} \mathbf{L} \]

\[ \frac{d \mathbf{L}}{dt} \parallel \mathbf{T} \]

but \( \mathbf{T} \) is horizontal
so \( d\mathbf{L} \) is perpendicular to \( \mathbf{g} \)

Also boomerangs, frisbees, satellites

Systems of particles
Total angular momentum \( \mathbf{L} \)
\[
\mathbf{L} = \sum (\mathbf{r}_i \times \mathbf{p}_i)
\]

\[
\frac{d}{dt} \mathbf{L} = \sum \frac{d}{dt} (\mathbf{r}_i \times \mathbf{p}_i)
= \sum \mathbf{T}_i
= \sum \mathbf{T}_i \text{ internal} + \sum \mathbf{T}_i \text{ external}
\]
Internal torques cancel in pairs (Newton 3)
\[
\therefore \sum \mathbf{T}_\text{ext} = \frac{d}{dt} \mathbf{L}
\]

\( \text{cf } F_{\text{ext}} = \frac{d}{dt} \mathbf{p} \)

where \( \sum \mathbf{T}_\text{ext} \) is the sum of all external torques.

(This equation derived for inertial frames but it is also true for other frames if centre of mass is taken as origin.)

Consequence:
If \( \sum \mathbf{T}_\text{ext} = 0 \), \( \frac{d}{dt} \mathbf{L} = 0 \).

Conservation of angular momentum of isolated system

Example Circular motion of ball on string. What happens to the speed of the ball as the string is shortened? (Neglect air resistance).

\[
\mathbf{L} = \mathbf{r} \times \mathbf{p}
= r p \sin \theta
= r m v \sin \theta
\]
Example: Person on rotating seat holds two 2.2 kg masses at arms' length. Draws masses in to chest.

What is the increase in $\omega$? Is K conserved?

Rough estimates: $k_{\text{person}}$ about long axis ~ 15 cm

$I_p = Mk^2 = \sim 70 \text{ kg} \cdot (0.15 \text{ m})^2$  
include moving part of chair

$I_p \sim 1.6 \text{ kgm}^2$

$I_m = \text{mr}^2 \equiv 2.2 \text{ kg} \cdot (0.8 \text{ m})^2$

$I_m \equiv 1.4 \text{ kgm}^2$  (arms extended)

$I_m' = \text{mr}^2 \equiv 2.2 \text{ kg} \cdot (0.2 \text{ m})^2$

$\equiv 0.1 \text{ kgm}^2$  (arms in)

No external torques $\Rightarrow L_i = L_f$

$(I_p + 2I_m)\omega_i = (I_p + 2I_m')\omega_f$

$\omega_f = \frac{I_p + 2I_m}{I_p + 2I_m'} \sim 2.4$

$K_f = \frac{1}{2} (I_p + 2I_m)\omega_f^2 = 2.4$

Arms do work: $Fds = ma_{\text{centrip}}ds$

Example  Space-walking cosmonaut ($m = 80 \text{ kg}$, $k = 0.3 \text{ m}$ about short axes) throws a 2 kg ball (from shoulder) at 31 ms$^{-1}$ ($\vec{v}$ displaced 40 cm from c.m.). How fast does she turn? Is this a record?

In orbit so no ext torques so $L$ conserved

$L_i = L_f = L_{\text{ball}} + L_{\text{cos}}$

$0 = \vec{r} \times m\vec{v} - I\omega$

$= rm\vec{v} - Mk^2\omega$

$\omega = \frac{mv}{Mk^2}$

$= 3.4 \text{ rad.s}^{-1}$

$= 33 \frac{1}{3} \text{ r.p.m.}$