Mechanics for systems of particles and extended bodies

PHYS1121-1131 UNSW. Session 1

Left: trajectory of end of rod. Right: parabola is the trajectory followed by the Centre of mass

In a finite body, not all parts have the same acceleration. Not even if it is rigid. How to apply \( \mathbf{F} = m \mathbf{a} \)?

Total mass \( M = \sum m_i \)

Define the centre of mass as the point with displacement \( \mathbf{r}_{\text{cm}} = \sum \frac{m_i \mathbf{r}_i}{M} \)

Why this definition? Consider \( n \) particles, \( m_i \) at positions \( \mathbf{r}_i \). \( \mathbf{F}_i \) acts on each. For each particle, N2 gives \( \mathbf{F}_i = m_i \mathbf{a}_i \). Add these to get total force acting on all particles:

\[
\sum \mathbf{F}_i = \sum m_i \mathbf{a}_i \quad \text{definition of acceleration}
\]

\[
= \sum m_i \frac{d^2}{dt^2} \mathbf{r}_i
\]

If masses constant, can change order of \( \frac{d}{dt} \) and multiply:

\[
= \sum \frac{d^2}{dt^2} m_i \mathbf{r}_i
\]

\[
= \frac{d^2}{dt^2} \sum m_i \mathbf{r}_i \quad \text{multiply top and bottom by } M
\]

\[
= M \frac{d^2}{dt^2} \left( \sum \frac{m_i \mathbf{r}_i}{M} \right)
\]

But we defined \( \mathbf{r}_{\text{cm}} = \frac{\sum m_i \mathbf{r}_i}{M} \). Then \( \sum \mathbf{F}_i = M \frac{d^2}{dt^2} \mathbf{r}_{\text{cm}} = M \mathbf{a}_{\text{cm}} \)

(total force) = (total mass) * (acceleration of centre of mass)
Look at forces in detail:

Each $\mathbf{F}_i$ is the sum of internal forces (from other particles in the body/system) and external forces (from outside the system)

$$\sum \mathbf{F}_i = \sum \mathbf{F}_{i,\text{internal}} + \sum \mathbf{F}_{i,\text{external}}$$

**Newton 3**: All internal forces $\mathbf{F}_{ij}$ between $i^{th}$ and $j^{th}$ particles are Newton pairs:

$$\mathbf{F}_{ji} = -\mathbf{F}_{ij}$$

$\therefore$ $\sum$ internal forces $= 0$

$\therefore \sum \mathbf{F}_i = \sum \mathbf{F}_{i,\text{external}} = \mathbf{F}_{\text{external}}$

$\therefore \mathbf{F}_{\text{external}} = \mathbf{M} \mathbf{a}_\text{cm}$

$$\left( \frac{\text{total external force}}{\text{mass}} \right) = \left( \frac{\text{total mass}}{\text{acceleration of centre of mass}} \right)$$
For $n$ discrete particles, **centre of mass** at

$$\mathbf{r}_{cm} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M} \quad (i)$$

For a continuous body, elements of mass $dm$ at $\mathbf{r}$

$$\mathbf{r}_{cm} = \frac{\int_V \mathbf{r} \, dm}{\int_V dm} = \frac{\int \mathbf{r} \, dm}{M} \quad (ii)$$  

This is the same equation. Really.

Can rearrange (i):

$$0 = \sum \frac{m_i \mathbf{r}_i - m_i \mathbf{r}_{cm}}{M} \quad -> \quad \sum m_i (\mathbf{r}_i - \mathbf{r}_{cm}) = 0$$

**(ii) ->** \(\int_{body} (\mathbf{r}_i - \mathbf{r}_{cm}) dm = 0\)

Later, when doing rotation, we'll consider which is a useful way to find c.m. experimentally. Three boys with equal mass $m$:  

![Diagram of three boys with equal mass m](image)
Example. Where is the c.m. of the earth moon system?

\[ r_{cm} = \frac{\sum m_i r_i}{\sum m_i} \]

Take origin at centre of earth.

\[ x_{cm} = \frac{m_e x_e + m_m x_m}{m_e + m_m} = \frac{m_m d}{m_e + m_m} = 4,600 \text{ km} \quad \text{i.e. inside the earth.} \]

recall: when doing Newton's 3rd, we derived the centre of rotation of this system

Example

On a square plate (mass \( m_p \)), we place \( m_1 \) and \( m_2 \) as indicated.

\( m_p = 135 \text{ g}, m_1 = 100 \text{ g} \) and \( m_2 = 50 \text{ g} \)

Where is the cm of the system?

\[ r_{cm} = \frac{\sum m_i r_i}{\sum m_i} \]

\[ = \frac{m_p(1.5\mathbf{i} + 1.5\mathbf{j}) + m_1(2.0\mathbf{j}) + m_2(2.0\mathbf{i} + 2.0\mathbf{j})}{m_p + m_1 + m_2} \]

\[ = \frac{(303\text{g}\mathbf{i} + 503\text{g}\mathbf{j})}{285\text{g}} = 1.1 \mathbf{i} + 1.8 \mathbf{j} \]

Check?

\[ \text{check that} \ \sum m_i (r_i - r_{cm}) = 0 \]
Example. Rod, cross-section A, made of length a of material with density $\rho_2$ and length b of material with density $\rho_1$. Where is c.m.?

If $\rho_1 = 2\rho_2$, and $a = 2b$, where is cm?

\[ r_{cm} = \frac{\int r \, dm}{\int dm} \]

How am I going to integrate $dm$ and over the whole body?

\[ dm = \rho dV = \rho dx \]

\[ x_{cm} = \frac{\int x \, dm}{\int dm} \]

Put origin at join and $\rho$ is constant for the integrations

\[
\begin{align*}
0 & \quad a \\
\int_0^{a} \rho_1 Ax \, dx + \int_0^{a} \rho_2 Ax \, dx \\
-\int_{-b}^{0} \rho_1 Ax \, dx + \int_{0}^{a} \rho_2 Ax \, dx \\
& = \frac{-\frac{1}{2} \rho_1 b^2 + \frac{1}{2} \rho_2 a^2}{\rho_1 b + \rho_2 a} \\
& = \frac{a^2 - rb^2}{2(a + rb)} \quad \text{where we define } r = \frac{\rho_1}{\rho_2}
\end{align*}
\]

Internal vs external work.

Problem. Skateboarder pushes away from a wall

Point of application of $\mathbf{F}$ does not move, \( \therefore \) normal force does no work, but K changes. Where does energy come from? \textit{Obvious: arms!}

\[ F_{ext} = Ma_{cm} \]

\[ F_{ext} \, dx = Ma_{cm} \, dx_{cm} = M \frac{dv_{cm} \, dx_{cm}}{dt} = Mv_{cm} \, dv_{cm} \]

"Centre of mass work"

\[ W_{cm} = \int_i^f F_{ext} \, dx = \left( \frac{1}{2} Mv_{cm}^2 \right)_f - \left( \frac{1}{2} Mv_{cm}^2 \right)_i \]

Work done = that which would have been done if $F_{ext}$ had acted on cm.
Momentum
Definition: \( p = mv \)

In relativity, we’ll find that this is a low \( v \) approximation to
\[
(p = \frac{mv}{\sqrt{1 - v^2/c^2}})
\]
and also that \( K = (\gamma - 1)mc^2 \)

Generalised form of

**Newton 2:**
\[
\Sigma F = \frac{d}{dt} p
\]
\[
\Sigma F = m \frac{d}{dt} v + v \frac{d}{dt} m
\]

If m constant, \( \Sigma F = ma \) but for the general case, use the general expression

**System of particles:** What is system? - you choose: draw a boundary around it.
\[
P = \Sigma p_i \quad \text{and} \quad M = \Sigma m_i
\]
\[
P = \Sigma m_i v_i = \Sigma m_i \frac{d}{dt} r_i
\]
\[
= \frac{d}{dt} \Sigma m_i r_i = M \frac{d}{dt} \left( \frac{\Sigma m_i r_i}{M} \right)
\]
\[
P = My_{cm}
\]

If M constant:
\[
\frac{d}{dt} P = Ma_{cm}
\]
\[
\Sigma F_i = \Sigma \frac{d}{dt} p_i = \frac{d}{dt} P
\]
All internal forces are in pairs \( F_{ji} = -F_{ij} \)

\[.\]
\[ F_{ext} = \frac{d}{dt} P \] two very important conclusions:

i) Motion of cm is like that of particle mass M at \( r_{cm} \) subjected to \( F_{ext} \).

ii) If \( F_{ext} = 0 \), momentum of whole system is conserved

Note that momentum is a vector, so we have a conservation law that can apply in one or more directions.
Example 90 kg man jumps \( (v_j = 5 \text{ ms}^{-1}) \) into a (stationary) 30 kg dinghy. What is their final speed? (Neglect friction.)

No external forces act in horizontal direction so \( P_x \) is conserved.

\[
P_i = P_f
\]

\[
\begin{align*}
man & \quad dinghy & \quad man & \quad dinghy \\
m_m v_j + 0 & = (m_m + m_d) v_f \\
\end{align*}
\]

\[
v_f = \frac{m_m}{m_m + m_d} v_j
\]

Example Rain falls into an open trailer (area 10 m\(^2\)) at 10 litres.min\(^{-1}\).m\(^{-2}\).

Neglecting friction, what \( F \) required to maintain constant speed of 10 ms\(^{-1}\)?

\[
F_x = \frac{d}{dt} (mv_x) = m \frac{d}{dt} v_x + v_x \frac{d}{dt} m
\]

\[
= 10 \text{ ms}^{-1} \times \left( \frac{10 \text{ kg.m}^{-2}}{60 \text{ s}} \right) \frac{10 \text{ m}^{-2}}{10 \text{ m}^2}
\]

\[
= 17 \text{ N}.
\]
**Example.** Rocket has mass $m = m(t)$, which decreases as it ejects exhaust at rate $r = -\frac{dm}{dt}$ and at relative velocity $u$. What is the acceleration of the rocket?

$$\left(\frac{dm}{dt} = \text{rate of increase of mass of rocket} < 0\right)$$

No external forces act so momentum conserved. In the frame of the rocket, forwards direction:

$$dp_{\text{rocket}} + dp_{\text{exhaust}} = 0$$

$$m \cdot dv + (-dm) \cdot (-u) = 0$$

$$dv = -\frac{dm}{m}$$

$$v = -u \frac{dm}{m}$$

$$a = \frac{dv}{dt} = -\frac{u}{m} \cdot \frac{dm}{dt}$$

$$a = \frac{ur}{m} \quad \text{1st rocket equation}$$

$$dv = -u \frac{dm}{m} = -u \frac{dm}{d(ln m)}$$

$$\int_{i}^{f} dv = v_{f} - v_{i} = u \ln \frac{m_{i}}{m_{f}} \quad \text{2nd rocket equation}$$

*need high exhaust velocity $u$ (c?), else require $m_{i} >> m_{f}$*
Collisions  Definition: in a collision, "large" forces act between bodies over a "short" time.

In comparison, we shall often neglect the momentum change due to external forces.

Example 1:

\[ F_{ab} = F_{ba} = 0 \]

forces that crumple cars during (brief) collision are much larger than friction force (tires - road), \( \therefore \) neglect \( F_{ext} \).

Be quantitative: suppose car decelerates from 30 kph to rest in a 20 cm 'crumple zone'.
Approximate as constant acceleration \( a = (v_f^2 - v_i^2)/2\Delta x = -170 \text{ m/s}^2 \), so force on car during collision \( \sim m|a| = 200 \text{ kN} \), compared with friction at \( \sim 10 \text{ kN} \).

Other safety questions while we are here: What is the force on occupants? If they have the same acceleration as the car, then \( F = ma \sim 10 \text{ kN} \): seat belts, air bags. What about reducing crumple zones? Forces on pedestrians? How big are crumple zones for pedestrians?
Example 2

Jupiter spacecraft doesn't "hit" Examples: deep space probes

Here, start and finish of collision not well defined
At large separation before and after, \( F_{ab} = F_{ba} \approx 0 \)
During collision (fly-by), forces are considerable.
However, \( F_{\text{grav}} \propto 1/r^2 \), so much smaller at large distances.

**Impulse (\( I \)) and momentum**

Newton 2 \( \Rightarrow \) \( \frac{dp}{dt} = F \)

\[ \therefore \quad \int_i^f dp = \int_i^f F \, dt \quad \text{so} \]

Definition: \( I = p_f - p_i = \int_i^f F \, dt \)

*In collisions, Impulse is integral of large internal force over short time*

Ball *is* inflated to normal pressure.
Can get an *under*estimate of force:

\[ F > \left( \text{pressure in ball} \right) \times \text{deformed area} \]

\[ \sim 70 \text{ kPa} \times 0.02 \text{ m}^2 \sim 1 \text{ kN} \]

Camera flashes at equal times

When is head of club travelling fastest?
Speed of ball compared to speed of club?
Usual case: external forces small, act for small time, therefore \[ \int_{i}^{f} F_{\text{ext}} \, dt \] is small.

\[ \Delta p_1 = \int_{i}^{f} F_1 \, dt = -F_1 \Delta t \]

\[ \Delta p_2 = \int_{i}^{f} F_2 \, dt = -\int_{i}^{f} F_1 \, dt \]

\[ \therefore \Delta p_1 = -\Delta p_2 \]

\[ \therefore \Delta P = \Delta p_1 - \Delta p_2 = 0 \]

**If external forces are negligible** (in any direction), then the momentum of the system is conserved (in that direction).

**Example.** Cricket ball, \( m = 156 \text{ g} \), travels at \( 45 \text{ m.s}^{-1} \). What impulse is required to catch it? If the force applied were constant, what average force would be required to stop it in \( 1 \text{ ms} \)? in \( 10 \text{ ms} \)? What stopping distances in these cases?

\[ v_i \quad \text{vi} \quad \text{v}_f = 0 \]

\[ m = 0.156 \text{ kg}, \quad v_i = 45 \text{ m.s}^{-1} \quad v_f = 0. \]

\[ I = p_f - p_i \]

\[ = m(v_f - v_i) \text{ to right} \]

\[ = .... = 7.0 \text{ kgm}^{-1} \text{ to left.} \]

\[ I = \int_{i}^{f} F \, dt \quad \text{if } F \text{ constant, } I = F\Delta t. \]

\[ \therefore F_{\text{av}} = I/\Delta t. \]

If const \( F \Rightarrow \) const a. \( s = v_{\text{av}}\Delta t \). \( v_{\text{av}} = 23 \text{ m.s}^{-1} \)

\[ \Delta t \quad 1.0 \text{ ms} \quad 10 \text{ ms} \]

\[ F \quad 7 \text{ kN} \quad 0.7 \text{ kN} \quad \text{ouch!} \]

\[ s \quad 2 \text{ cm} \quad 20 \text{ cm} \]
Example. (A common method to measure speed of bullet.) Bullet (m) with $v_b$ fired into stationary block (M) on string. (i) What is their (combined) velocity after the collision? (ii) What is the kinetic energy of the bullet? (iii) of the combination? (iv) How high does the block then swing?

Note the different stages and three diagrams:

a-b): collision, no horizontal external forces. : momentum conserved. Friction does work, so mechanical energy is lost, not conserved

b-c): during this phase, external forces do act, so momentum is lost, not conserved. However, there are no non-conservative forces, so mechanical energy conserved.

Analyse a) to b)

No horizontal ext forces during collision. : momentum conserved

i) \[ P_{xi} = P_{xf} \]
\[ m v_b = (m + M) v_t \]
\[ v_t = \frac{m}{m + M} v_b \]

ii) \[ K_b = \frac{1}{2} m v_b^2 \]

iii) \[ K_t = \frac{1}{2} (m + M) v_t^2 = \frac{1}{2} (m + M) \left( \frac{m}{m + M} v_b \right)^2 \]
\[ = \frac{1}{2} \frac{m^2}{m + M} v_b^2 < K_b. \]

Conclusion: \[ U_i = U_f, \quad K_i \neq K_f. \]

Mechanical energy is not conserved - deformation of block is not elastic; heat is produced.

Let’s look in more detail: little digression about elastic and inelastic collisions
During a collision with negligible external forces,

$$P = (\sum m) \mathbf{v}_{cm}$$ is conserved

$\sum m$ constant $\therefore \mathbf{v}_{cm}$ is constant $\therefore \frac{1}{2} M\mathbf{v}_{cm}^2$ constant

K of c.m. is not lost. But the K of components with respect to c.m. can be lost.

Greatest possible loss of K: if all final velocities = $\mathbf{v}_{cm}$, i.e. if all objects stick together after collision. Called completely inelastic collision.

**Completely inelastic collision.**

Contrast:
**Completely elastic collision** is one in which non-conservative forces do **no** work, so mechanical energy is conserved.

**Inelastic collision** is one in which non-conservative forces do some work, so mechanical energy is **not** conserved.

**Completely inelastic collision** is one in which all kinetic energy with respect to the centre of mass is lost: Non-conservative forces do as much work as possible, so as much mechanical energy as possible is lost

Part (iv) of previous example (b-c):

here the external forces (gravity and tension) **do** do work and change momentum. But there is no non-conservative force and so in **this part of the process** conservation of mechanical energy applies:

$$\Delta U + \Delta K = 0$$

$$(M + m)g (\Delta h - 0) + (0 - K_t) = 0$$

$$\Delta h = ... = \frac{1}{2} \frac{m^2}{g(m + M)^2} v_b^2$$

so we can rearrange and get $v_b$ from $\Delta h$
Example Elastic collision in one dimension

\[ v_{1i} \quad \rightarrow \quad v_{2i} \quad (\text{bang}) \quad \rightarrow \quad v_{1f} \quad \rightarrow \quad v_{2f} \]

Collision: neglect external forces ⇒

\[ p_i = p_f \]
\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad (i) \]

elastic ⇒ \[ K_i = K_f \]
\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (ii) \]

\[ \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \]

usually know \( m_1, m_2, v_{1i}, v_{2i} \). Two unknowns \( v_{1f}, v_{2f} \), \( \therefore \) we can always solve.

Or: transform to frame where (e.g.) \( v_1 = 0 \) \quad can simplify the algebra

Or: transform to centre of mass frame.

Example. Take \( m_1 = m_2, v_{2i} = 0, v_{1i} = v \).

\[ v \quad \rightarrow \quad v_{2i} = 0 \quad \rightarrow \quad v_{1f} \quad \rightarrow \quad v_{2f} \]

neglect external forces ⇒ \( p_i = p_f \)
\[ m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \quad (i) \]
\[ \frac{1}{2} m_1 v_{1i}^2 + 0 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \quad (ii) \]

(i) \[ \rightarrow \quad v_{2f} = v - v_{1f} \quad (iii) \]

\[ \frac{1}{2} m v^2 + 0 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m v_{2f}^2 \]

\[ (i) \rightarrow \quad v_{2f} = v - v_{1f} \quad (iii) \]

\[ \frac{1}{2} m v^2 + 0 = \frac{1}{2} m v_{1f}^2 + \frac{1}{2} m(v^2 + v_{1f}^2 - 2v v_{1f}) \]
\[ \therefore \quad 0 = v_{1f}^2 - vv_{1f} \]
\[ 0 = v_{1f} (v_{1f} - v) \quad 2 \text{ solutions} \]

Either: \quad \( v_{1f} = 0 \) and (iii)\[ (v_{1f} = 0) \rightarrow \quad v_{2f} = v \]

i.e. 1st stops dead, all p and K transferred to \( m_2 \)
or: \quad \( v_{1f} = v \) and (iii)\[ (v_{1f} = v) \rightarrow \quad v_{2f} = 0 \quad \text{i.e. missed it.} \]
Example  Show that, for an elastic collision in one dimension, the relative velocity is unchanged.

\[ v_{1f} - v_{2i} = v_{2f} - v_{1f} \]

\( p \) and \( K \) conservation gave:

(i) \( m_1(v_{1f} - v_{1i}) = -m_2(v_{2f} - v_{2i}) \)

(ii) \( \frac{1}{2} m_1(v_{1f}^2 - v_{1i}^2) = -\frac{1}{2} m_2(v_{2f}^2 - v_{2i}^2) \)

If they hit, \( (v_{1f} - v_{1i}) \neq 0, \text{  } (v_{2f} - v_{2i}) \neq 0 \) use \( a^2 - b^2 = (a - b)(a + b) \)

(ii)/(i) \( \Rightarrow \) \[ v_{1i} + v_{1f} = v_{2i} + v_{2f} \]

\[ v_{1i} - v_{2i} = v_{2f} - v_{1f} \]

i.e. relative velocity the same before and after

\[ \text{Solve} \rightarrow \quad v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i} + \frac{m_2 - m_1}{m_2 + m_1} v_{2i} \]

Example  Two similar objects, mass \( m \), collide completely inelastically.

case 1: \( v_{1i} = v, \text{  } v_{2i} = 0. \)

case 2: \( v_{1i} = v, \text{  } v_{2i} = -v. \)

What energy is lost in each case?

\( p \) conserved \( \Rightarrow \) \[ mv_{1i} + mv_{2i} = 2mv_f \]

\[ v_f = \frac{v_{1i} + v_{2i}}{2} \]

\[ \Delta K = K_f - K_i = \frac{1}{2} (2m) v_f^2 - \frac{1}{2} mv_{1i}^2 - \frac{1}{2} mv_{2i}^2 \]

case 1: \[ \Delta K = \frac{1}{2} (2m) \left( \frac{v + 0}{2} \right)^2 - \frac{1}{2} mv^2 \]

\[ = -\frac{1}{4} mv^2 \]

case 2: \[ \Delta K = \frac{1}{2} (2m) \left( \frac{0 + 0}{2} \right)^2 - \frac{1}{2} mv^2 - \frac{1}{2} mv^2 \]

\[ = -mv^2 \quad 4 \text{ times as much energy lost} \]

Remember this if you have the choice in traffic, rugby etc
Elastic collisions in 2 (& 3) dimensions

Choose frame in which $m_2$ stationary, $v_{1i}$ in x dir

$b$ is called impact parameter (distance "off centre")

$p_x$ conserved \[ m_1 v_{1i} = m_{v1f} \cos \theta_1 + m_{v2f} \cos \theta_2 \]

$p_y$ conserved \[ 0 = m_{v2f} \sin \theta_2 - m_{v1f} \sin \theta_1 \]

$K$ conserved

\[
\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_{v1f}^2 + \frac{1}{2} m_{v2f}^2 + \Delta K \quad (iii)
\]

where $\Delta K = 0$ for elastic case

3 equations in $v_{1f}, v_{2f}, \theta_1$ and $\theta_2$: need more info (often given $\theta_1$ or $\theta_2$)

Incidentally: for hard spheres, neglecting rotation and friction (reasonable during collision, but not after)

\[(R + R) \sin \theta_2 = b \quad \theta_2 = \sin^{-1} \left( \frac{b}{2R} \right) \]

\[
\begin{array}{c|c}
\theta_2 & \frac{b}{R} \\
\hline
90^\circ & 2 \\
1 & 1
\end{array}
\]

i) Note that as $\theta \to 90^\circ$, small error in $b$ gives large error in $\theta_2$.

ii) Experiment on billiard table: Does $b = R$ give $\theta_2 = 30^\circ$?
Example. Police report of road accident. Car 1, mass m₁ strikes stationary car m₂ at point C. They then slide to rest in positions shown. Given μ_k = μ (assumed same for both) find the initial speed v of m₁. Can you check assumption? (real example)

After collision, a for both = \( \frac{F_f}{m} = -\mu \frac{W}{m} = -\mu g \)

\[
\begin{align*}
v_f^2 - v_i^2 &= 2as = -2\mu gs \\
0 - v_1^2 &= -2\mu g s_1 \\
v_1 &= \sqrt{2\mu g s_1} \\
v_2 &= \sqrt{2\mu g s_2}
\end{align*}
\]

Neglect external forces during collision: \( \Delta P = 0 \)

\( P_x: \) \( m_1v = m_1v_1 \cos \theta + m_2v_2 \cos \phi \) (i)

\( P_y: \) \( 0 = m_1v_1 \sin \theta - m_2v_2 \sin \phi \) (ii)

(i) \( \Rightarrow \) \( v = \sqrt{2\mu g s_1} \cos \theta + (m_2/m_1) \sqrt{2\mu g s_2} \cos \phi \)

Note the "spare" equation—we can use it to check the model or assumptions:

(ii) \( \Rightarrow \) \( m_1\sqrt{2\mu g s_1} \sin \theta = m_2\sqrt{2\mu g s_2} \sin \phi \)

\[
\frac{\mu_2}{\mu_1} = \frac{s_1 m_1^2 \sin^2 \theta}{s_2 m_2^2 \sin^2 \phi}
\]

(The \( \mu \) may not be the same for the two: surfaces different, orientation of wheels etc)
Balloonist Albert writes message on a bottle (1 kg) and drops it over the side. It is falling vertically at 40 m.s\(^{-1}\) when caught by parachutist Zelda (m = 50 kg), travelling at 1 m.s\(^{-1}\) at 45° to vertical. Collision (bottle—Zelda's hand) lasts 10 ms.

i) If only gravity acted, what is \(\Delta p\) for Zelda over 10 ms?

ii) Neglecting ext forces during collision, what is the velocity of (Zelda+bottle) after collision?

iii) What impulse is applied to bottle during collision?

iv) What is the impulse applied to Zelda?

v) What is the average force during collision?

vi) Will Albert and Zelda live happily ever after?

\(\text{i) due to } \mathbf{W} , \Delta \mathbf{p} = \mathbf{W} \Delta t = .. = 5 \text{ kgm.s}^{-1} \text{ down}\)

\(\text{ii) Neglect ext forces } \Rightarrow \text{ momentum conserved.}\)

\[m_b \mathbf{v}_{bi} + m_Z \mathbf{v}_{zi} = m_{(Z+b)} \mathbf{v}_{(Z+b) f}\]

\[1(-40 \mathbf{j}) + 50(1 \cos 45° \mathbf{i} - 1 \cos 45° \mathbf{j}) = 51(v_x \mathbf{i} + v_y \mathbf{j})\]

\(i\) \_ dim: \(v_x = \cos 45° \cdot \frac{50}{51} \times 1 = 0.7 \text{ ms}^{-1}\)

\(j\) \_ dim: \(v_y = -\frac{1 \times 40 - 50 \cos 45°}{51} = -1.5 \text{ ms}^{-1}\)

\(\therefore |v_f| = \sqrt{v_x^2 + v_y^2} = 1.6 \text{ ms}^{-1}\)

\(\theta_f = \tan^{-1} \frac{v_y}{v_x} \Rightarrow 67° \text{ to horizontal}\)

\(\text{iii)} \quad \mathbf{I}_b = \mathbf{p}_{bf} - \mathbf{p}_{bi} = 1x(v_x \mathbf{i} + v_y \mathbf{j}) - 1(-40 \mathbf{j}) = (1.6 \mathbf{i} + 38 \mathbf{j}) \text{ kgm.s}^{-1}\)

\(\text{iv)} \quad \mathbf{I}_Z = -\mathbf{I}_b = -(1.6 \mathbf{i} + 38 \mathbf{j}) \text{ kgm.s}^{-1}\)

\(|\mathbf{I}_Z| = \sqrt{1.6^2 + 38^2} = 38 \text{ kgm.s}^{-1}\)

\(\text{v)} \quad \mathbf{F}_Z = \frac{\Delta \mathbf{p}_Z}{\Delta t} = \frac{|\mathbf{I}_Z|}{\Delta t} = .. = 380 \text{ N}\)
**Example** Controlled demolition. A building has $S$ stories, each of height $h$. Explosions destroy the strength of the $n^{th}$ floor. How long before $(n+1)^{th}$ floor hits ground, falling vertically?

Assume that the building remains intact above the explosion and inelastic collisions with the lower floors. To obtain a lower bound, assume negligible strength between lower floors.

$(S-n)$ floors have mass $(S-n)m$. To get the lower estimate on falling time, assume no strength in the demolished floor, $s$ the upper floors are in free fall (as a rigid body) for a distance $h$ with acceleration $g$. So they strike the next floor with speed $v_n = \sqrt{2gh}$.

Inelastic collision with next floor gives speed $v$ where:

$$(S-n)mv_n = (S-n+1)mv$$

Let the falling mass after any collision have initial speed $v_0$ and speed before the next collision be $v_c$.

$$v_c^2 - v_0^2 = 2gh$$

$$v_c = \sqrt{2gh + v_0^2} = v_0\left(1 + \frac{2gh}{v_0^2}\right)^{1/2}$$

For $i^{th}$ collision

$$(S-n+i-1)mv_{c(i)} = (S-n+i)mv_{0(i+1)}$$

$$v_{c(i)} = \sqrt{2gh + (v_0(i))^2}$$

$$\frac{S-n+i-1}{S-n+i} \sqrt{2gh + (v_0(i))^2} = v_0(i+1)$$

$$v_0(i+1) = \left(1 - \frac{1}{S-n+i}\right)\sqrt{2gh + (v_0(i))^2}$$

$$h = v_0t + \frac{1}{2}gt^2$$

$$0 = \frac{1}{2}gt^2 + v_0t - h$$

$$t = -\frac{v_0 \pm \sqrt{v_0^2 + 2gh}}{g}$$