Gravity

Notes for PHYS 1121-1131. Joe Wolfe, UNSW

Gravity: where does it fit in?

<table>
<thead>
<tr>
<th>Gravity [general relativity]</th>
<th>Electric force*</th>
<th>Weak nuclear force</th>
<th>Strong nuclear force</th>
<th>Colour force</th>
</tr>
</thead>
<tbody>
<tr>
<td>gravitons</td>
<td>photons</td>
<td>intermediate vector bosons</td>
<td>pions</td>
<td>gluons</td>
</tr>
</tbody>
</table>

- electro-weak

Grand Unified Theories

Some tries for classical gravity

Theories Of Everything

* Electromagnetism "unified" by Maxwell, and also by Einstein: Magnetism can be considered as the relativistic correction to electric interactions which applies when charges move.

- Only gravity and electric force have macroscopic ("infinite") range.

  \[ m_{graviton} = m_{photon} = 0 \]

- Gravity weakest, but dominates on large scales because it is always attractive

Greeks to Galileo:

i) things fall to the ground ("natural" places)

ii) planets etc move (variety of reasons)

but no connection (in fact, natural vs supernatural)

Newton's calculation:

accel\(n\) of moon

\[
\frac{r_m}{r_{moon}}
\]

\[
\begin{align*}
= r_m \omega_m^2 \\
= (3.8 \times 10^8 \text{ m}) \left( \frac{2\pi}{27.3 24 3600} \right)^2 \\
= 2.7 \times 10^{-3} \text{ m.s}^{-2}
\end{align*}
\]

accel\(n\) of "apple" = 9.8 m.s\(^{-2}\)

\[
\frac{a_{apple}}{a_{moon}} = 3600; \quad \frac{r_m}{R_e} = \frac{385000 \text{ km}}{6370 \text{ km}} = 60; \\
\left( \frac{r_m}{R_e} \right)^2 = 3600
\]

Newton's brilliant idea: What if the apple and the moon accelerate according to the same law? → What if every body in the universe attracts every other, inverse square law?
Newton's law of gravity:

\[ F = -G \frac{m_1 m_2}{r^2} \]

Negative sign means \( F // -r \)

Why is it inverse square? Wait for Gauss' law in electricity.

\[ F_{12} = -F_{21} \]

Newton already knew Kepler's empirical law:

For planets, \( r^3 \propto T^2 \) orbit radius and period

Now if \( F \propto a c \propto \frac{1}{r^2} \)

then constant = \( a c r^2 = r\omega^2 r^2 = r^3 \omega^2 \)

<table>
<thead>
<tr>
<th>Planet</th>
<th>( r ) from sun</th>
<th>( T )</th>
<th>( \omega )</th>
<th>( r\omega^2 )</th>
<th>( r^3 \omega^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mercury</td>
<td>58</td>
<td>7.62</td>
<td>8.25 (10^{-7} )</td>
<td>3.95 (10^{-5} )</td>
<td>1.31 (10^{20} \text{m}^3\text{s}^{-2} )</td>
</tr>
<tr>
<td>Venus</td>
<td>108</td>
<td>19.4</td>
<td>3.23 (10^{-7} )</td>
<td>1.13 (10^{-5} )</td>
<td>1.32 (10^{20} \text{m}^3\text{s}^{-2} )</td>
</tr>
<tr>
<td>Earth</td>
<td>150</td>
<td>31.6</td>
<td>1.99 (10^{-7} )</td>
<td>5.94 (10^{-6} )</td>
<td>1.33 (10^{20} \text{m}^3\text{s}^{-2} )</td>
</tr>
<tr>
<td>etc</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
How big is G?

Cavendish's experiment (1798)

\[ F = -G \frac{m_1 m_2}{r^2} \]

From deflection and spring constant, calculate F, know \(m_1\) and \(m_2\), \(\therefore\) can calculate G. \(G = 6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2} \) (or \(m^3\text{kg}^{-1}\text{s}^{-2}\))

Now also weight of \(m\): \(|W| = mg = G \frac{mM_e}{R_e^2}\)

\(\therefore\) Cavendish first calculated mass of the earth:

\[ M_e = \frac{gR_e^2}{G} = \frac{9.8 \text{ m.s}^{-2} \times (6.37 \times 10^6 \text{ m})^2}{6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}} = 6.0 \times 10^{24} \text{ kg} \]


Some numbers

What is force between two oil tankers at 100 m?

\[ F = -G \frac{m_1 m_2}{r^2} \]

What happens when more there are \(\geq 3\) bodies?

Superposition principle.

\[ \mathbf{F}_{\text{all objects together}} = \sum \mathbf{F}_{\text{individual}} \]

or \[ \mathbf{F}_1 = \sum \mathbf{F}_{1i} \]

force on \(m_1\) due to masses \(m_i\)

continuous body \[ \mathbf{F}_1 = \int_{\text{body}} \mathbf{d} \mathbf{F} \]
Shell theorem

A uniform shell of mass M causes the same gravitational force on a body outside is as does a point mass M located at the centre of the shell, and zero force on a body inside it.

\[ F_g = \frac{GMm}{R^2} \]

Example. If \( \rho_{\text{earth}} \) were uniform (it isn’t), how long would it take for a mass to fall through a hole through the earth to the other side?

\[ M_r = \rho \frac{4}{3} \pi r^3 \]

\[ \therefore F_r = -G \frac{m \rho \frac{4}{3} \pi r^3}{r^2} \]

\[ F = -Kr \quad \text{where} \quad K = Gm \rho \frac{4}{3} \pi \]

\[ \therefore \text{motion is SHM with} \quad \omega = \sqrt{\frac{K}{m}} \quad \text{Simple Harmonic Motion: discussed later} \]

\[ T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{G\rho \frac{4}{3} \pi}} = \frac{2\pi}{\sqrt{GM/R^3}} = \ldots = 84 \text{ minutes} \]

\[ \therefore \text{falls through (one half cycle) in 42 minutes} \quad (\text{actually faster for real density profile}) \]
Gravity near Earth's surface

\[ W = \left| F_g \right| = G \frac{M m}{R_e^2} \]
\[ W = mg_o = G \frac{M_e m}{r^2} \]

\( g_o \) is acceleration in an inertial (non-rotating) frame

\[ g_o = G \frac{M_e}{r^2} \]

Usually, \( r \approx R_e \), but

\[ g_o = G \frac{M_e}{(R_e + h)^2} = g_s \left( \frac{R_e}{R_e + h} \right)^2 \]
\[ = g_s \left( \frac{1}{1 + h/R_e} \right)^2 \]

where \( g_s \) is \( g_o \) at surface

Other complications:

i) Earth is not uniform (especially the crust) *useful for prospecting*

ii) Earth is not spherical

iii) Earth rotates (see Foucault pendulum)

(Weight) \( = - \) (the force exerted by scales)

At poles, \( F - N = 0 \)

At latitude \( \theta \), \( F - N = ma \)

where \( a = r \omega^2 = (R_e \cos \theta) \omega^2 \)

\[ = .... = 0.03 \text{ ms}^{-2} \text{ at equator} \]
\[ = 0 \text{ at poles} \]

We define \( -g = \frac{N}{m} = \frac{F - ma}{m} \)

So \( g \) is greatest at the poles, least at the equator, and does not (quite) point towards centre.

horizontal \( \perp g \)

Earth is flattened at poles
**Puzzle:** How far from the earth is the point at which the gravitational attractions towards the earth and that towards the sun are equal and opposite? Compare with distance earth-moon (380,000 km)

\[ |F_e| = |F_s| \]
\[ \frac{GM_em}{d^2} = \frac{GM_sm}{(r_e - d)^2} \]
\[ M_e(r_e - d)^2 = M_sd^2 \]
\[ r_e^2 - 2r_e d + d^2 = \frac{M_s}{M_e} d^2 \]
\[ \left( \frac{M_s}{M_e} - 1 \right) d^2 + 2r_e d - r_e^2 = 0 \]
\[ d = \ldots = ? \]

**Gravitational field.** A field is ratio of force on a particle to some property of the particle. For gravity, (gravitational) mass is the property:

\[ \frac{F_{grav}}{m} = g = g(r) \]

is a vector field

cf electric field \[ \frac{F_{elec}}{q} = E(r) \] later in syllabus

**Gravitational potential energy.** Revision:

**Potential energy**

For a conservative force \( F \) (i.e. one where work done against it, \( W = W(r) \)) we can define potential energy \( U \) by \( \Delta U = W_{against} \) i.e.

\[ \Delta U = - \int F \cdot dr \]

near Earth's surface, \( F_g = mg \equiv \text{constant} \)

\[ = - \int (-mgk) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \]

\[ = mg k \cdot k \int_i^f dz = mg (z_f - z_i) \]

choose reference at \( z_i = 0 \), so \( U = mgz \)

**Gravitational potential energy** of \( m \) and \( M \).

\[ \Delta U_g = - \int F_g \cdot ds = \int F_g dr = \int G \frac{Mm}{r^2} dr = -GMm \left[ \frac{1}{r_f} - \frac{1}{r_i} \right] \]

**Convention:** take \( r_i = \infty \) as reference: \( U(r) = -\frac{GMm}{r} \)

\( U = \text{work to move one mass from} \infty \text{to} r \text{ in the field of the other. Always negative.} \)

Usually one mass >> other, we talk of \( U \) of one in the field of the other, but it is \( U \) of both.
Escape "velocity".
Escape "velocity" is minimum speed \(v_e\) required to escape, i.e. to get to a large distance \((r \to \infty)\).

![Diagram](attachment:image.png)

Projectile in space: no non-conservative forces so conservation of mechanical energy

\[
\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0 + 0
\]

\[
v_{esc} = \sqrt{\frac{2GM}{R}}
\]

For Earth: \(v_{esc} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2} \times 5.98 \times 10^{24} \text{ kg}}{6.37 \times 10^6 \text{ m}}}\)

\[
= 11.2 \text{ km.s}^{-1} = 40,000 \text{ k.p.h.}
\]

Put launch sites near equator: \(v_{eq} = R_e\omega_e = 0.47 \text{ km.s}^{-1}\)

**Question** In Jules Verne's "From the Earth to the Moon", the heros' spaceship is fired from a cannon*. If the barrel were 100 m long, what would be the average acceleration in the barrel?

\[
v_f^2 - v_i^2 = 2as
\]

\[
a = \frac{v_e^2 - 0}{2s} = \frac{(1.12 \times 10^4 \text{ ms}^{-2})^2}{2 \times 100 \text{ m}}
\]

\[
= 630,000 \text{ ms}^{-2} = 64,000 \text{ g}
\]

* why? If you burn all the fuel on the ground, you don't have to accelerate and to lift it. Much more efficient.

**Planetary motion**

"The music of the spheres" - Plato

Leucippus & Democritus C5 B.C.

heliocentric universe

Hipparchus (C2 BC) & Ptolemy (C2 AD) geocentric universe

Tycho Brahe (1546-1601) - very many, very careful, naked eye observations.

Johannes Kepler joined him. He fitted the data to these empirical laws:

**Kepler's laws:**

1. All planets move in elliptical orbits, with the sun at one focus.
   
   *Except for Pluto and Oort cloud objects, these ellipses are \(\equiv\) circles.
   
   \(M_{sun} \gg m_{planet}\), so sun is \(\equiv c.m.\)

2. A line joining the planet to the sun sweeps out equal areas in equal time.
   
   *Slow at apogee (distant), fast at perigee (close)*

3. The square of the period \(\propto\) the cube of the semi-major axis
   
   *Slow for distant, fast for close*
Newton’s explanations:

Law of areas:

![Diagram of areas](image)

Area = \( \frac{1}{2} r_r \delta \theta \)

i.e. for same \( \delta t \), \( \frac{1}{2} r^2 \delta \theta = \) constant

Conservation of angular momentum \( L \). Sun at c.m.

\[ L = \lvert \mathbf{r} \times \mathbf{p} \rvert = \lvert \mathbf{r} \times m v \rvert = m r v_{\text{tangential}} \]

\[ = m r \omega = m r^2 \frac{\delta \theta}{\delta t} \]

\[ = \frac{m}{\delta t} r^2 \delta \theta = \text{constant.} \]

Conservation of \( L \) \( \Rightarrow \) Kepler 2.

Law of periods: (we consider only circular orbits)

Kepler 3: \( T^2 \propto r^3 \)

Newton 2 \( \rightarrow \)

\[ F = m a = m r \omega^2 \]

\[ G \frac{Mm}{r^2} = m \left( \frac{2\pi}{T} \right)^2 \]

\[ T^2 = \left( \frac{4\pi^2}{GM} \right) r^3 \rightarrow \text{Kepler 3} \]

(works for ellipses with semi-major axis \( a \) instead of \( r \))

Newton 2 & Newton’s gravity \( \Rightarrow \) Kepler 3

Newton 2 & Newton’s gravity also \( \Rightarrow \) Kepler 1

Example What is the period of the smallest earth orbit? (\( r = R_e \))

What is period of the moon? (\( r_{\text{moon}} = 3.82 \times 10^8 \) m)

\[ T_1 = \sqrt{\frac{4\pi^2}{GM} r^3} = \sqrt{\frac{4 \pi^2}{6.67 \times 10^{-11} \times 5.98 \times 10^{24} \times (6.37 \times 10^6)^3}} \text{ s} \]

\[ = 84 \text{ min} \]

Kepler 3: \( T^2 \propto r^3 \)

\[ \frac{T_2}{T_1} = \left( \frac{r_2}{r_1} \right)^{3/2} = \left( \frac{3.82 \times 10^8}{6.37 \times 10^6} \right)^{3/2} = 464 \]

\[ T_2 = 464 \times T_1 = 27.2 \text{ days} \]

For other planets: most have moons, so the mass of the planet can be calculated from

\[ T^2 = \left( \frac{4\pi^2}{GM} \right) r^3 \]
Orbits and energy

No non-conservative forces do work, so mechanical energy is constant:

\[
E = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}
\]

Let's remove \( v \). Consider circular orbit:

\[
\frac{v^2}{r} = a_c = \frac{F}{m} = \frac{GMm}{r^2m}
\]

\[
\therefore \quad \frac{1}{2}mv^2 = \frac{1}{2} \frac{GMm}{r}
\]

\[
E = K + U = \frac{1}{2} \frac{GMm}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}
\]

i.e. \( E = \frac{1}{2}U \), or \( K = -\frac{1}{2}U \), or \( K = -E \).

Small \( r \Rightarrow U \) very negative, \( K \) large. \( \text{(inner planets fast, outer slow)} \)

Example

A spacecraft in orbit fires rockets while pointing forward. Is its new orbit faster or slower?

\[ \mathbf{F} \parallel \mathbf{ds} \therefore \text{Work done on craft} \]

\[ W = \int \mathbf{F} \cdot d\mathbf{s} > 0. \]

\[
\therefore E = -\frac{GMm}{2r} \text{ increases, i.e. it becomes less negative. (R is larger). } K = -E, \therefore K \text{ smaller, so it travels more slowly. } \]

\( \text{called "Speeding down"} \)

Quantitatively:

\[
K_i = -E_i, \quad K_f = -E_f = -(E_i + \Delta E)
\]

\[
K_f = K_i - \Delta E
\]

\[
\frac{1}{2}mv_i^2 = \frac{1}{2}mv_f^2 - W
\]

\( \text{Looks odd, but need lots of work to get to a high, slow orbit.} \)
Manœuvring in orbit.

To catch up, vessel 1 fires engines *backwards*, and loses energy. It thus falls to a lower orbit where it travels faster, until it catches up. It then fires its engines *forwards* in order to slow down (it climbs back to the original, slower orbit).

**Example:** In what orbit does a satellite remain above the same point on the equator?

*Called the Clarke Geosynchronous Orbit*

i) Period of orbit = period of earth's rotation
   \[ T = 23.9 \text{ hours} \]
   \[ T^2 = \left( \frac{4\pi^2}{GM} \right) r^3 \]
   \[ r = \sqrt[3]{\frac{GM T^2}{4\pi^2}} = \ldots \]
   \[ = 42,000 \text{ km} \quad \text{popular orbit!} \]