Newton's laws:
  - force, mass, acceleration also weight
Friction - coefficients of friction
Hooke's Law
Dynamics of circular motion

Aristotle: \( v = 0 \) is "natural" state
Galileo & Newton: \( a = 0 \) is "natural" state

**Newton's Laws**

**First** "zero (total) force \( \Rightarrow \) zero acceleration"

more formally:

If \( \Sigma \mathbf{F} = 0 \), \( \exists \) reference frames in which \( a = 0 \)
\( \exists = \) "there exists"
called **Inertial frames**
observation: w.r.t. these frames, distant stars don't accelerate

In inertial frames:

**Second** \( \Sigma \mathbf{F} = m \mathbf{a} \)
\( \Sigma \) is important

\( (\Sigma F_x = ma_x \quad \Sigma F_y = ma_y \quad \Sigma F_z = ma_z) \)
3D \( \rightarrow \) 3D equations
1st law is special case of 2nd

What does the 2nd law mean?
\( \Sigma \mathbf{F} = m_i \mathbf{a} \quad \text{and} \quad \mathbf{W} = m_g \mathbf{g} \)
are \( m_i \) and \( m_g \) necessarily the same?
called inertial and gravitational masses

\( \mathbf{F} = m \mathbf{a} \)
\( \mathbf{a} \) is already defined.

i) Does this equation define \( m \)?
ii) Does this law define \( \mathbf{F} \) ?
iii) Is it a physical law?
iv) All of the above?
v) How?
Newton 1: "Every body persists in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed on it."

postulate

An inertial frame of reference is one in which Newton's 1st law is true.

definition

Such frames exist (and w.r.t. these frames, distant stars don't accelerate)

observation:

∴ if \( \Sigma F = 0 \), \( a = 0 \) w.r.t. distant stars.

Force causes acceleration, \( \mathbf{F} \parallel \mathbf{a}, \mathbf{F} \propto \mathbf{a} \)

definition

Newton 2: To any body may be ascribed a scalar constant, mass, such that the acceleration produced in two bodies by a given force is inversely proportional to their masses,

i.e. for same \( F \), \( \frac{m_2}{m_1} = \frac{a_1}{a_2} \)

Already have metre, second, choose a standard body for kg, then choose units of \( F \) (Newtons) such that

\[ \mathbf{F} = m \mathbf{a} \]

(this eqn. is laws 1&2, definition of mass and units of force)

Newton 3: "To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal and directed to contrary parts"

Or

Forces always occur in pairs, \( \mathbf{F} \) and \( -\mathbf{F} \), one acting on each of a pair of interacting bodies.

Important conclusion: internal forces in a system add to zero.

\[ \mathbf{F}_{AB} \quad \text{m}_A \quad \text{m}_B \quad \mathbf{F}_{BA} \]

Third \( \mathbf{F}_{AB} = -\mathbf{F}_{BA} \)

Why so?

\[ \mathbf{F}_{AB} \quad \text{m}_A \quad \text{m}_B \quad \mathbf{F}_{BA} \]

\[ \text{m}_A \quad \text{m}_B \quad \mathbf{F}_{AB} + \mathbf{F}_{BA} \]
**Example** Where is centre of earth-moon orbit?

\[ F_e = F_m = F_g \] equal and opposite  

each makes a circle about common centre of mass

\[ F_g = m_m a_m = m_m \omega^2 r_m \]

\[ F_g = m_e a_e = m_e \omega^2 r_e \]

\[ \frac{r_m}{r_e} = \frac{m_e}{m_m} = \frac{5.98 \times 10^{24} \text{ kg}}{7.36 \times 10^{22} \text{ kg}} = 81.3 \] (i)

earth-moon distance \( r_e + r_m = 3.85 \times 10^8 \text{ m} \) (ii)

solve \[ r_m = 3.80 \times 10^8 \text{ m}, \quad r_e = 4.7 \times 10^6 \text{ m} \]

\[ \therefore \text{ centre of both orbits is inside earth (later: c. of mass)} \]

**Example.** As the bus takes a steady turn with radius 8 m at constant speed, you notice that a mass on a string hangs at 30° to the vertical. How fast is the bus going?

*Draw a diagram with physics*

mass in circular motion with bus, so force on \( m \):

\[ F_{\text{horiz}} = ma = m \frac{v^2}{r} \]

Only the tension has a horizontal component, so

\[ T \sin 30^\circ = m \frac{v^2}{r} \]

Need one more eqn: mass is not falling down, ie

vertical acceleration = 0, so

\[ T \cos 30^\circ = mg \]

Eliminate \( T \):

\[ \tan 30^\circ = m \frac{\sqrt{gr}}{r} \frac{1}{mg} \]

\[ \rightarrow v = \sqrt{gr \tan 30^\circ} = 6.7 \text{ m/s} = 24 \text{ kph} \rightarrow 20 \text{ kph} \text{ to 1 sig fig} \]

Check dimensions. Reasonable? And so...?
**Problem.** Horse and cart. Wheels roll freely.

Why should I pull? The force of the cart on me equals my force on it, but opposes it. \( \Sigma F = 0 \). We'll never accelerate.

Let's go!

Horizontal forces on cart (mass \( m_c \))

\[
F_c = m_c a_c = m_c a
\]

Horizontal forces on horse (mass \( m_h \))

\[
F_g - F_c = m_h a
\]

Horizontal forces on Earth (mass \( m_E \))

\[
m_E >> m_h + m_c
\]
"light" ropes etc.

Here, light means $m \ll$ other masses

Truck ($m_t$) pulls wagon ($m_w$) with rope ($m_r$).
All have same $a$.

\[ F_2 = m_w a \]  
\[ -F_1 - F_{ext} = m_t a \]  
\[ F_1 - F_2 = m_r a \]  
\[ -F_2 = m_w a \]  
\[ F_1 = F_2 \]  

Forces at opposite ends of light ropes etc are equal and opposite.

**Example.** Train. Wheels roll freely. Loco exerts horizontal force $F$ on the track. What are the tensions $T_1$ and $T_2$ in the two couplings?

Whole train accelerates together with $a$.

*Look at the external forces acting on the train (horiz. only).*

\[ F = (m+m+m)a \]  
\[ a = F/3m \]

*Look at horiz forces on car 2:*

\[ T_2 = ma = F/3 \]

*and on cars 2 and 1 together*

\[ T_1 = 2ma \]
Hooke's Law.

No applied force  
\( (x = 0) \)

Under tension  
\( x > 0 \)

\( \mathbf{F}_{\text{spring}} \) in opposite direction to \( x \).  
Experimentally, \( |\mathbf{F}_x| \approx |x| \) over small range of \( x \)

\[ F = -kx \]

**Hooke's Law.**

*linear elastic behaviour - more in S2.*

Why linear elasticity?

Intermolecular forces \( F \) and energies \( U \):

See tut problem on interatomic forces
Mass and weight
(inertial) mass \( m \) defined by \( F = ma \)

**observation:**
near earth's surface and without air,
all (?) bodies fall with same \( a \) (\( \approx -g \))
weight \( W = mg \)

**Warning:** do not confuse mass and weight, or their units

<table>
<thead>
<tr>
<th>Unit</th>
<th>Meaning</th>
<th>Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td>kg</td>
<td>mass</td>
<td></td>
</tr>
<tr>
<td>N</td>
<td>force</td>
<td>(kg.m.s(^{-2}))</td>
</tr>
<tr>
<td>kg wt</td>
<td>weight of 1 kg</td>
<td>mg = 9.8 N</td>
</tr>
</tbody>
</table>

What is your weight?

*Why is \( W \propto m \)?*  *Why is \( mg \propto m_i \)?*

\( ma = F = W = (\text{Grav field}).(\text{grav. property of body}) \)
* Mach's Principle
* Principle of General Relativity
* Interactions with vacuum field
Example Grav. field on moon $g_m = 1.7 \, \text{ms}^{-2}$. An astronaut weighs 800 N on Earth, and, while jumping, exerts 2kN while body moves 0.3 m. What is his weight on moon? How high does he jump on earth and on moon?

$$mg_E = W_E \rightarrow m = \frac{800 \, \text{N}}{9.8 \, \text{ms}^{-2}} = 82 \, \text{kg}$$

$$W_m = mg_m = 82 \, \text{kg} \times 1.7 \, \text{ms}^{-2} = 140 \, \text{N}$$

Vertical (y) motion with const accel. While feet are on ground,

$$\Sigma F = 2 \, \text{kN} - W_E$$

$$= 1.2 \, \text{kN} \text{ (Earth)}$$

Moon:

$$\Sigma F = 2 \, \text{kN} - mg_m = 1.9 \, \text{kN}$$

Jump has two parts:

- feet on ground \( a = \frac{\Sigma F}{m} \) \( v_i = 0, v_f = v_j \)
- feet off ground \( a = -g \) \( v_i = v_j, v_f = 0 \)

While on ground:

$$v_j^2 - v_0^2 = 2a\Delta y = 2 \frac{\Sigma F}{m} \Delta y$$

Earth $\rightarrow v_j = 3.0 \, \text{ms}^{-1}$ Moon $\rightarrow v_j = 3.7 \, \text{ms}^{-1}$

While above ground:

$$v^2 - v_j^2 = -2gh \rightarrow h = \frac{v_j^2}{2g}$$

$$h_E = 0.5 \, \text{m} \quad h_m = 4 \, \text{m}$$
Example
Light pulley, light string. What is acceleration of the masses?

Let $a$ be accel (down) of $m_1 =$ accel (up) of $m_2$.
Newton 2 for $m_1$: $T - m_1 g = - m_1 a$
- $m_2$: $T - m_2 g = + m_2 a$
subtract: 
$- m_1 g + m_2 g = - m_1 a - m_2 a$
\[ a = \frac{m_1 - m_2}{m_1 + m_2} g \]

(Check: if $m_1 = m_2$, $a = 0$. If $m_2 = 0$, $a = g$.)

Contact forces
The normal component of a contact force is called the **normal force $N$**. The component in the plane of contact is called the **friction force $F_f$**.

**Normal force**: at right angles to surface, is provided by deformation.
- If $\exists$ relative motion, **kinetic friction** (whose direction opposes relative motion)
- If $\exists$ no relative motion, **static friction** (whose direction opposes applied force)

**Define** coefficients of kinetic ($k$) and static ($s$) friction:
\[ |F_f| = \mu_k N \quad |F_f| \leq \mu_s N \]
Friction follows this **approximate** empirical law
\[ \mu_s \text{ and } \mu_k \text{ are approx. independent of } N \text{ and of contact area.} \]

Often $\mu_k < \mu_s$.

(It takes less force to keep sliding than to start sliding.)
Example. $\theta$ is gradually increased to $\theta_c$ when sliding begins. What is $\theta_c$? What is $a$ at $\theta_c$?

Newton 2 in normal dirn:
\[ N - mg \cos \theta = 0 \quad (i) \]

Newton 2 in dirn down plane:
\[ mg \sin \theta - F_f = ma. \quad (ii) \]

No sliding:  $a = 0$

\[ \therefore (ii) \Rightarrow mg \sin \theta = F_f \leq \mu_s N \quad (i) \Rightarrow \mu_s mg \cos \theta \]

\[ mg \sin \theta \leq \mu_s mg \cos \theta \]

\[ \tan \theta \leq \mu_s, \quad \theta_c = \tan^{-1}\mu_s \quad \text{useful technique for } \mu_s \]

Sliding at $\theta = \theta_c$:  $a > 0$

\[ \therefore (ii) \Rightarrow a = g \sin \theta_c - \frac{F_f}{m} \]

\[ = g \sin \theta_c - \frac{\mu_k N}{m} \quad (i) \Rightarrow \]

\[ = g \sin \theta_c - \mu_k g \cos \theta_c \]

\[ = g \cos \theta_c (\mu_s - \mu_k) \]
Example. Rear wheel drive car, 300 kg wt on each front wheel, 200 kg wt on rear.
Rubber-road:
\( \mu_s = 1.5, \mu_k = 1.1 \)

Neglect rotation of car during accelerations. Assume that brakes produce 1.8 times as much force on front wheels as on back.

(i) What is max forward accel without skidding? What is maximum deceleration (ii) not skidding? (iii) 4 wheel skid?

\[
F_{frs} \leq \mu_s N_R = \mu_s W_R
\]
\[
= 1.5 \times 200 \times 9.8 \text{ N} = 2.9 \text{ kN}
\]
\[
a_{\text{max}} = \frac{F}{m} = \frac{2 \times 2.9 \text{ kN}}{(2 \times 300 + 2 \times 200) \text{ kg}} = 5.8 \text{ ms}^{-2}
\]

Stopping.
For all wheels, \( F_{fs} \leq \mu_s N = \mu_s W \)
\[F_{ff} = 1.8 F_{fr} \]
\[\mu_s W_F = 1.5 \mu_s W_R, \]
\[
\text{front wheels skid first, when } F_{ff} > \mu_s W_F.
\]
max total friction = (front + rear) = \( 2 + \frac{2}{1.8} \) \( \mu_s W_F \)
\[
= 1400 \text{ kg.wt} = 14 \text{ kN}
\]
(ii) \( a_{\text{max}} = \frac{F_{\text{max}}}{m} = \frac{14 \text{ kN}}{1000 \text{ kg}} = 14 \text{ ms}^{-2} \)
(iii) \( a = \frac{\sum F_k}{m} = \frac{\sum \mu_k W}{m} = ... = 11 \text{ ms}^{-2} \)

Questions:
Does area of rubber-road contact make a difference?
Does the size of the tire make a difference?
**Example**
Conical pendulum
(Uniform circular motion.)
What is the frequency?

*Apply Newton 2 in two directions:*
Vertical: \( a_y = 0 \) \( \therefore \Sigma F_y = 0 \)
\( \therefore T \cos \theta - W = 0 \)
\( T = \frac{mg}{\cos \theta} \)

Horizontal:
\[
\frac{mv^2}{r} = ma_c = T \sin \theta = \frac{mg \sin \theta}{\cos \theta}
\]
\( \therefore \frac{v^2}{r} = g \tan \theta \)
\( \therefore v = \sqrt{rg \tan \theta} \)
\( \therefore \frac{2\pi r}{T} = \sqrt{rg \tan \theta} \)
\( \therefore f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{g \tan \theta}{r}} \)
Example.

Apply force $F$ at $\theta$ to horizontal. Mass $m$ on floor, coefficients $\mu_s$ and $\mu_k$. For any given $\theta$, what $F$ is required to make the mass move?

Eliminate 2 unknowns $N$ and $F_f \rightarrow F(\theta, \mu_s, m, g)$

Stationary if 
$$F_f \leq \mu_s N \quad (1)$$

Newton 2 vertical: 
$$N = mg + F \sin \theta \quad (2)$$

Newton 2 horizontal: 
$$F \cos \theta = F_f \quad (3)$$

(1,3) $\rightarrow$ stationary if 
$$F \cos \theta \leq \mu_s N$$

$$F \cos \theta \leq \mu_s (mg + F \sin \theta) \quad \text{(using (2))}$$

$$F (\cos \theta - \mu_s \sin \theta) \leq \mu_s mg \quad (*)$$

Note importance of sign of $(...)$

if 
$$\cos \theta - \mu_s \sin \theta = 0,$$

$$\theta = \theta_{\text{crit}} = \tan^{-1}(1/\mu_s).$$

If $\theta < \theta_c$, then $\cos \theta - \mu_s \sin \theta > 0$

stationary if 
$$F \leq \frac{\mu_s mg}{\cos \theta - \mu_s \sin \theta}$$

i.e. moves when 
$$F > F_{\text{crit}} = \frac{\mu_s mg}{\cos \theta - \mu_s \sin \theta}$$

If $\theta > \theta_c$, then $\cos \theta - \mu_s \sin \theta < 0$ 

$(*) \Rightarrow$ stationary if 
$$F \geq \frac{\mu_s mg}{\cos \theta - \mu_s \sin \theta}$$

i.e. stationary no matter how large $F$ becomes.
Centripetal acceleration and force
Circular motion with $\omega = \text{const.}$ and $v = \text{const.}$, e.g., bus going round a corner

Consider hammer thrower

Resultant force produces acceleration in the horizontal direction, towards the centre of the motion

**Centripetal force, centripetal acceleration**

**Example** Plane travels in horizontal circle, speed $v$, radius $r$. For given $v$, what is the $r$ for which the normal force exerted by the plane on the pilot = twice her weight? What is the direction of this force?

Centripetal force $F = m \frac{v^2}{r} = N \cos \theta$

Vertical forces: $N \sin \theta = mg$

Eliminate $\theta$: $N^2 = m^2 \left( \frac{v^4}{r^2} + g^2 \right)$

$$\left( \frac{N^2}{m^2} - g^2 \right) = \frac{v^4}{r^2} \rightarrow r = \sqrt{\frac{v^2}{\sqrt{\frac{N^2}{m^2} - g^2}}}$$

$$\sin \theta = \frac{mg}{N} = \frac{1}{2}$$

$\therefore$ $30^\circ$ above horizontal, towards axis of rotation
**Example.** Foolhardy lecturer swings a bucket bricks in a vertical circle. How fast should he swing so that the bricks stay in contact with the bucket at the top of the trajectory?

*Draw diagram & identify important variables
pose question mathematically.*

**W** and **N** provide centripetal force.

\[ mg + N = ma_c \]

For contact, we need

\[ N \geq 0 \]

so \( ma_c \geq mg \) *how to express \( a_c \)?*

\[
a_c = \frac{v^2}{r} = rw^2 = r\left(\frac{2\pi}{T}\right)^2
\]

*is easy to measure*

\[ T = 2\pi \sqrt{\frac{r}{a_c}} \leq 2\pi \sqrt{\frac{r}{g}}
\]

\[ r \sim 1\text{ m} \rightarrow T \leq 2\text{ s.}
\]

**Question.** Three identical bricks. What is the minimum force you must apply to hold them still like this?

Vertical forces on middle brick add to zero:

\[ 2F_f = mg \]

Define \( \mu_s \)

\[ F_f \leq \mu_s N \]

\[ \therefore N \geq \frac{F_f}{\mu_s} = \frac{mg}{2\mu_s} \]

Bricks not accelerating horizontally, so normal force from hands = normal force between bricks.

\[ \therefore \text{(each) hand must provide} \geq \frac{mg}{2\mu_s} \text{ horizontally.} \]

Vertically, two hands together provide 3mg.