Work and Energy  
(PHYS 1121 & 1131, UNSW, Session 1, 2011)  
- (the dot product)  
- definition of work  
- definition of kinetic energy  
- restatement of Newton 2  
- conservative and non-conservative forces  
- potential energy  

Sometimes, the physics sense of work is very like the use in normal language. This bloke is doing work

\[ \text{Mechanics > Energy and power > } 7.1 \text{ dW} = F \cdot ds = Fds \cos \theta \]

\[ \text{force newtons} \]

\[ \text{height metres} \]

\[ \text{Force} \quad \text{Height} \]

\[ \text{force newtons} \]

\[ \text{height metres} \]

\[ \text{Force} \quad \text{Height} \]

\[ \text{work joules} \]

\[ s \text{ and F at right angles, no work done} \]

*but the trolley isn’t doing work. Why not?*  See Physclips, Work and Energy
We need some new maths: The scalar product. 

Why? e.g. Work: scalar, related to $\mathbf{F}$, $\mathrm{d} s$ and $\theta$.

dot product

because it makes maths easier

$\mathbf{F}$

\[ \mathrm{d}W = |\mathbf{F}| |\mathrm{d} s| \cos \theta \]

(later: also used for voltage $\mathrm{d}V = |\mathbf{E}| |\mathrm{d} s| \cos \theta$ etc)

therefore define

\[ \mathbf{a} \cdot \mathbf{b} = ab \cos \theta = (\mathbf{b} \cdot \mathbf{a}) \]

pronounced "a dot b"

Apply to unit vectors:

\[ \mathbf{i} \cdot \mathbf{i} = 1 \cdot 1 \cos 0^\circ = 1 = \mathbf{i} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} \]

\[ \mathbf{i} \cdot \mathbf{j} = 1 \cdot 1 \cos 90^\circ = 0 = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} \]

Scalar product by components

\[ \mathbf{a} \cdot \mathbf{b} = (a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k}) \cdot (b_x \mathbf{i} + b_y \mathbf{j} + b_z \mathbf{k}) \]

\[ = (a_x b_x) \mathbf{i} \cdot \mathbf{i} + (a_y b_y) \mathbf{j} \cdot \mathbf{j} + (a_z b_z) \mathbf{k} \cdot \mathbf{k} \]

\[ + (a_x b_y + a_y b_x) \mathbf{i} \cdot \mathbf{j} + (a_z b_x - a_x b_z) \mathbf{k} \cdot \mathbf{i} \]

where \[ \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1 \]

and these terms are all zero, so

\[ \mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \]

which is an important result

And, at no extra charge, we get a useful geometrical tool

Problem. Find the angle between

\[ \mathbf{a} = 4 \mathbf{i} - 3 \mathbf{j} + 7 \mathbf{k} \]

\[ \mathbf{b} = 2 \mathbf{i} + 5 \mathbf{j} - 3 \mathbf{k} \]

Hooee! Imagine doing this by geometry. Let's use dot product

which, using the result above, we can two ways:

\[ ab \cos \theta = \mathbf{a} \cdot \mathbf{b} = a_x b_x + a_y b_y + a_z b_z \]

\[ \cos \theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}} = \frac{4*2 - 3*5 - 7*3}{\sqrt{4^2 + 3^2 + 7^2} \sqrt{2^2 + 5^2 + 3^2}} \]

\[ = \ldots \]

\[ \Rightarrow \theta = 122^\circ \]

hit the calculator to give:
Is it easier for the sailor to climb the mast using the halyard (a rope passing through a pulley at the top of the mast)?

Why?

Neglecting acceleration:

Without rope:
\[ W = F_{\text{feet}} + F_{\text{hands}} \]

With rope:
\[ W = F_{\text{feet}} + F_{\text{hands}} + T \]

but \( F_{\text{hands}} = T \) so
\[ W = F_{\text{feet}} + 2F_{\text{hands}} \]

During the moment when \( F_{\text{feet}} = 0 \), your hands apply 50% less force! But how do you "pay for" the reduction in force? Let's introduce work.

**Definition of work**

When force varies, use differential displacement \( ds \)

\[ dW = F \, ds \cos \theta = \mathbf{F} \cdot \mathbf{ds} \]

we can think of this in two ways:

\[ (F) (ds \cos \theta) \rightarrow F \times \text{component of } ds \parallel F, \text{ or} \]
\[ (F \cos \theta) (ds) \rightarrow ds \times \text{component of } F \parallel ds \]

\[ W = \int_{0}^{L} F \cos \theta \, ds \]

if \( F \) and \( \theta \) are constant, we get \( W = FL \cos \theta \)

SI Unit: 1 Newton x 1 metre = 1 Joule

But this is the baby version: forces do vary!
SIMPLE MACHINES (pulleys, levers, screws, inclined planes etc)

Example. How much work is done by lifting 100 kg vertically by 1.8 m very slowly?

\[
\text{Slow} \therefore F_{\text{applied}} \cong mg \\
W = mg \cdot d \cos 0^\circ = mgh : \text{more later} \\
= 1.8 \text{ kJ.}
\]

Not a lot – how much if you walk up one flight of stairs?
Yet it is harder to do, because the force is inconveniently large. Consider:

If the rope and pulleys are light, and if the accelerations are negligible, then
Force on LH pulley
\[
ma = 0 = 2T - mg \\
\therefore T = mg/2
\]

If mass rises by D, work done = mgD.
But rope shortens on both sides of rising pulley,
if mass rises by D, rope must be pulled 2D, so
work done = T \cdot 2D = mgD

We do the same work with less force by covering more distance.

Example. What is the work done by gravity in a circular orbit?

\[
W = \int F \cdot ds \cos \theta \\
= 0
\]

Historically important: no work to do!
**Example.** \( F_{\text{grav}} \propto \frac{1}{r^2} \). How much work is done to move \( m = 1 \) tonne from earth's surface \((r = 6500 \text{ km})\) to \( r = \infty \)?

\[
W = \int F \, ds \cos \theta
\]

\[
= \int F \, dr
\]

\[
F = -F_{\text{grav}} = \frac{Cm}{r^2}
\]

more later, but for now, what is the constant \( C \)? What do we know?

On earth's surface, we've dropped objects so we know that \( a = F/m = -9.8 \text{ ms}^{-2} \)

\[
\therefore C = (9.8 \text{ ms}^{-2})(6.5 \times 10^6 \text{ m})^2 = 4.1 \times 10^{14} \text{ m}^3\text{s}^{-2}
\]

\[
W = \int_{6500 \text{ km}}^{\infty} \frac{Cm}{r^2} \, dr
\]

note: potential energy proportional to \(-\frac{1}{r}\)

\[
= -Cm \left( \frac{1}{\infty} - \frac{1}{6.5 \times 10^6} \right)
\]

not equal to \( mgh \). More on this later

\[
= 6.3 \times 10^{10} \text{ J} = 63 \text{ GJ}.
\]

Worse: rockets very inefficient: as we'll see later

**Work to deform spring**

No applied force

\((x = 0)\)

Hooke's law:

\[
\text{Work done by spring } = \int F_{\text{spring}} \, dx
\]

\[
= \int -kx \, dx = -\frac{1}{2} kx^2 + 0
\]

Work done on spring = \( \int F_{\text{applied}} \, dx \)

\[
= \int kx \, dx = \frac{1}{2} kx^2
\]

(= work stored in spring)
The work-energy theorem

(Total) force $F$ acts on mass $m$ in $x$ direction.

\[
\begin{array}{c}
\text{i} \quad v_i \\
\text{f} \quad v \\
\end{array}
\]

\[
F = F_{\text{total}}
\]

\[
\begin{array}{c}
\text{i} \quad \text{Work done by } F = \int_i^f F \, dx \\
\text{f} \quad \text{(use } F = ma) \\
\end{array}
\]

\[
\begin{align*}
\text{i} & \quad \int_i^f m \frac{dv}{dt} \, dx = \int_i^f m \frac{dx}{dt} \, dv \\
\text{f} & \quad \int_i^f m \cdot v \, dv = \left[ \frac{1}{2} mv^2 \right]_i^f \\
\end{align*}
\]

\[
\text{Work done by } F = \frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2 = \Delta K
\]

Define kinetic energy $K = \frac{1}{2} mv^2$

Increase in kinetic energy of body = work done by total force acting on it.

\[
\text{This is a theorem, ie a tautology} \\
because it is only true by definition of KE and by Newton 2. \\
\therefore \text{restatement of Newton 2 in terms of energy. Not a new law}
\]

Work energy theorem (baby version)

Mechanics > Energy and power > 7.2 The Work energy theorem
doubling the speed $v \rightarrow 2v$ would give $K \rightarrow 4K$ four times as much kinetic energy

so same braking force must act over 4 times the distance

**Power.** is the rate of doing work

$$\bar{P} = \frac{W}{\Delta t}$$

$$P = \frac{dW}{dt}$$

SI unit: 1 Joule per second $= 1$ Watt (1 W)

**Example**  Jill (m = 60 kg) climbs the stairs in Matthews Bldg and rises 50 m in 1 minute. How much work does she do against gravity? What is her average output power? (neglect accelerations)

$$W = \int F \cdot ds = \int F_y dy \quad \text{(only y displacement matters, because mg acts in (-ve) y direction)}$$

$$F_y = mg$$

$$W = mg \int dy = mg \Delta y$$

$$= 29 \text{ kJ} \quad \text{(cf } K = \frac{1}{2} mv^2 \sim 20 - 40 \text{ J)}$$

$$\bar{P} = \frac{W}{\Delta t} = \frac{mg \Delta y}{\Delta t} = 490 \text{ W}$$

(to give a scale, humans can produce 100s of W, car engines several tens of kW)

$$1 \text{ horsepower} = 550 \text{ ft.lb.s}^{-1} = 0.76 \text{ kW}$$

**Potential energy.**

e.g. Compress spring, do W on it, but get no K. Yet can get energy out: spring can expand and give K to a mass. $\rightarrow$ Idea of stored energy.

e.g. Gravity: lift object (slowly), do work but get no K. Yet object can fall back down and give back K.
Recall \( W_{\text{against grav}} = mg \Delta y \) i.e. \( W = W(y) \)

**But:** Slide mass slowly along a surface. Do work against friction, but can't recover this energy mechanically. Not all forces "store" energy. Look at these three diagrams:

For the spring and gravity, when we change the direction of the displacement the force *doesn't* change direction, so the sign of the work done changes, so, round a closed path, the work done is zero.

For friction, when we change the direction of the displacement the force *does* change direction, so the sign of the work done doesn't change, so friction does negative work, and we do positive work against it.

So we have two very different sorts of forces.
Conservative and non-conservative forces

W against gravity \( \neq - \int_{i}^{f} F_{g} \, dr \cos \theta \)
\( \neq - \int_{i}^{f} F_{z} \, dz \)
\( \neq mg \int_{i}^{f} dz \)
\( \neq mg (z_{f} - z_{i}) \) \( \text{in uniform field} \)

W is uniquely defined at all \( r \), i.e. \( W = W(r) \)
If \( z_{f} - z_{i} \) are the same, \( W = 0 \).

\( \therefore \) Work done against gravity round a closed path = 0  
This is the definition

Gravity is a conservative force

Spring

W against spring \( \neq - \int_{i}^{f} F_{\text{spring}} \, dx \)
\( \neq - \int_{i}^{f} -kx \, dx \)
\( \neq \frac{1}{2} k(x_{f}^{2} - x_{i}^{2}) \)
W is uniquely defined at all \( x \), i.e. \( W = W(x) \)
\( x_{f} = x_{i} \Rightarrow W = 0 \).

\( \therefore \) Work done round a closed path = 0

Spring force is a conservative force

Friction

dW against fric \( \neq - F_{f} \, ds \cos \theta \)
but \( F_{f} \) always has a component opposite \( ds \)

\( \therefore \) dW always \( \geq 0 \). \( \text{(we never get work back)} \)

\( \therefore \) cannot be zero round closed path, \( \therefore W \neq W(r) \)

\( \therefore \) friction is a non-conservative force

Note that direction of friction (dissipative force) is always against motion. Direction of g doesn't change.
Potential energy
For a conservative force $F$ (i.e. one where work done against it, $W = W(r)$) we can define potential energy $U$ by $\Delta U = W$ against. i.e.

$$\Delta U = - \int_i^f F \, dr \cos \theta$$

Same examples: spring

$$\Delta U_{\text{spring}} = - \int_i^f F_{\text{spring}} \, dx$$

$$= \frac{1}{2} k(x_f^2 - x_i^2)$$

Choice of zero for $U$ is arbitrary.
Here $U = 0$ at $x = 0$ is obvious, so

$$U_{\text{spring}} = \frac{1}{2} kx^2$$

From energy to force:

$$U = - \int F \, ds \quad \text{where } ds \text{ is in the direction } // F$$

$$F = - \frac{dU}{ds}$$

in fact $F_x = - \frac{dU}{dx}$, $F_y = - \frac{dU}{dy}$, $F_z = - \frac{dU}{dz}$

Spring: $U_{\text{spring}} = \frac{1}{2} kx^2 \quad \therefore \quad F_{\text{spring}} = -kx$

Gravity: $U_g = mgz \quad \therefore \quad F_g = - \frac{dU}{dz} = -mg$

Energy of interaction:
repulsive

attractive

Hooke’s law

total force

repulsive

attractive

total energy

Hooke’s law

repulsive

attractive

"r"
Energy diagrams and equilibria:

Treat this as $y(x)$ for a particle in a uniform gravitational field, we can see $U(x)$ and imagine the direction of force ($-\frac{dU}{dx}$).

Minima give stable equilibria: stable with respect to small perturbations. Maxima give unstable equilibria.

Similar energy diagrams in chemistry and elsewhere.

**Conservation of mechanical energy** (sometimes!)

Recall: Increase in $K$ of body = work done by total force acting on it. (restatement of Newton 2)

But, if all forces are conservative, work done by these forces $= -\Delta U$ (definition of $U$)

∴ if only conservative forces act, $\Delta K = -\Delta U$

We define mechanical energy $E \equiv K + U$

so, if only conservative forces act, $\Delta E = 0$.

*we can make this stronger.*

Work done by non-conservative forces

Define internal energy $U_{\text{int}}$ where

$\Delta U_{\text{int}} = -\text{Work done by n-c forces}$

$(= +\text{Work done against n-c forces})$

Recall def'n of $K$: $\Delta K = \text{work done by } \Sigma \text{force}$

∴ $\Delta K = -\Delta U - \Delta U_{\text{int}}$

∴ $\Delta K + \Delta U + \Delta U_{\text{int}} = 0$

If n-c forces do no work, then $\Delta U_{\text{int}} = 0$, so:

**If non-conservative forces do no work,**

$\Delta E = \Delta K + \Delta U = 0$

or: **mechanical energy $E$ is conserved**

*Equivalent to Newton 2, but useful for many mechanics problems where integration is difficult.*

*State the principle carefully!*

**Never, ever write:** "kinetic energy — potential energy"

3 reasons why not: It's not true. In general, it gives the wrong answer. It makes examiners angry.
**Classic problem.** Child pushes off with \( v_i \). How fast is the s/he going at the bottom of the slide? Neglect friction (a non-conservative force).

\[ \Sigma F \]

\[ \rightarrow v \]

\[ \begin{aligned}
\text{i)} \quad & \text{By Newton 2 directly:} \\
& v = \int_{\text{top}}^{\text{bottom}} a \, dt = \int_{\text{top}}^{\text{bottom}} \frac{F}{m} = \int_{\text{top}}^{\text{bottom}} g \cos \theta \, dt = \ldots.
\end{aligned} \]

\[ \begin{aligned}
\text{ii)} \quad & \text{Using work energy theorem (Newton 2 indirectly):} \\
& \text{Non-conservative forces do no work, .} \therefore \text{mechanical energy is conserved, i.e.}
\end{aligned} \]

\[ \begin{aligned}
\Delta E = \Delta K + \Delta U &= 0 \\
K_f - K_i + U_f - U_i &= 0 \\
or \\
K_f + U_f &= K_i + U_i
\end{aligned} \]

\[ \begin{aligned}
\frac{1}{2} mv_f^2 - \frac{1}{2} mv_i^2 + mgy_f - mgy_i &= 0
\end{aligned} \]

\[ \begin{aligned}
\text{rearrange -> } v_f = \sqrt{v_i^2 + 2g(y_f - y_t)}
\end{aligned} \]

**Conservation of energy**

*observation:* for many forces, \( W = W(r) \), i.e. the work done by or against these forces is a function only of position. Therefore, for these forces only, it’s useful to define \( U = U(r) \).

*observation:* for all systems yet studied, \( U_{\text{int}} \) is a state function, i.e. \( U_{\text{int}} = U_{\text{int}}(\text{measured variables}) \)

Hence idea of internal energy. e.g.:

*Friction, \( -U_{\text{int}} \) = heat produced when work is done against friction.*

*Air resistance \( -U_{\text{int}} \) is sound and heat.*

*Combustion engines and animals: \( +U_{\text{int}} \) comes from chemical energy*

\[ \Delta K + \Delta U + \Delta U_{\text{int}} = 0 \]

is statement of Newton 2 plus definitions of \( K, U, U_{\text{int}} \).

The statement that \( \Delta U_{\text{int}} \) is a state function is the first law of thermodynamics. It is a law, i.e falsifiable. More on this in Heat.
Example. Freda (m = 60 kg) rides pogo stick (m << 60 kg) with spring constant k = 100 kN.m\(^{-1}\). Neglecting friction, how far does spring compress if jumps are 50 cm high?

Non-conservative forces do no work, \(\therefore\) mechanical energy is conserved, i.e.

\[ E_{\text{bottom}} = E_{\text{top}} \]

\[ K_b + U_b = K_t + U_t \quad \quad \quad \quad (U = U_{\text{grav}} + U_{\text{spring}}) \]

\[ \frac{1}{2} \, mv_{\text{horiz}}^2 + (mgy_b + \frac{1}{2} \, kx_b^2) \approx \frac{1}{2} \, mv_{\text{horiz}}^2 + (mgy_t + \frac{1}{2} \, kx_t^2) \]

\[ mg(y_t - y_b) \approx \frac{1}{2} \, kx_b^2 \]

\(\therefore\) \(x_b \approx \sqrt{\frac{2mg(y_t - y_b)}{k}} \quad \text{substitute} \quad \approx 80 \, \text{mm}.\)

Example. Slide starts at height \(h_1\). Later there is a hump with height \(h_2\) and (vertical) radius \(r\). What is the minimum value of \(h_2 - h_1\) if slider is to become airborne? Neglect friction, air resistance.

Over hump, \(a_c = \frac{v^2}{r} \) (down). Airborne if \(g < a_c\), i.e. if \(v_2^2 > gr\).

No non-conservative forces act so

\[ E_2 = E_1 \]

\[ U_2 + K_2 = U_1 + K_1 \]

\[ mgy_2 + \frac{1}{2} \, mv_2^2 = mgy_1 + \frac{1}{2} \, mv_1^2 \]

\[ \frac{1}{2} \, mv_2^2 = mg(y_1 - y_2) \]

\( (y_1 - y_2) = \frac{v_2^2}{2g} > \frac{gr}{2g} = \frac{r}{2} \)
**Example**  Bicycle and rider (80 kg) travelling at 20 m.s\(^{-1}\) stop without skidding. \(\mu_s = 1.1\). What is minimum stopping distance? How much work done by friction between tire and road? Between brake pad and rim? Wheel rim is ~300 g with specific heat ~ 1 kJ.kg\(^{-1}\), how hot does it get?

*friction \rightarrow\) *deceleration* \rightarrow* stopping distance*

\[
|a| = \frac{F_f}{m} \leq \frac{\mu_s N}{m} = \mu_s g
\]

\[
|a| \leq \mu_s g
\]

\[
v_f^2 - v_i^2 = 2as \quad \Rightarrow \quad s = \frac{v_f^2 - v_i^2}{2a}
\]

\[
s \geq \left| \frac{-v_i^2}{2\mu_sg} \right| = 19 \text{ m}
\]

Work done by friction between tire and road?

No skidding, \(\therefore\) no relative motion, \(\therefore\) W = 0.

Between pad and rim? Here there is relative motion.

All K of bike & rider \(\rightarrow\) heat in rim and pad

\[
W = \Delta K = K_f - K_i = -16 \text{ kJ}
\]

\[
Q = mC\Delta T \quad \text{.....}
\]

\[
\Delta T \sim 50 ^\circ C \quad (Heat\ and\ this\ definition\ come\ later\ in\ the\ syllabus)
\]
Example  Which way is it easier to drag an object?

\[
\begin{align*}
\text{horizontal: } F \cos \theta &= F_f \\
\text{vertical: } N + F \sin \theta &= mg
\end{align*}
\]

Suppose we move at steady speed, \( a = 0 \). Which requires less \( F \)? Which requires less work?

\[
\begin{align*}
\text{mechanical equilibrium} &\rightarrow \\
\text{horizontal: } F \cos \theta &= F_f \\
\text{vertical: } N + F \sin \theta &= mg
\end{align*}
\]

sliding \( \rightarrow \) Ff = \( \mu_k \)N

\[
F \cos \theta = \mu_k N
\]

eliminate N \( \rightarrow \)

\[
F \cos \theta = \mu_k (mg - F \sin \theta)
\]

\[
F = \frac{\mu_k mg}{\cos \theta + \mu_k \sin \theta}
\]

when \( \theta = 0 \), \( F = \mu_k mg \)

\( F < F' \) if \( \cos \theta + \mu_k \sin \theta > 1 \), i.e. if \( \mu_k \) large & \( \theta \) small

Work done = Fs \( \cos \theta = F_f s \)

\[
= \mu_k Ns = \mu_k s(mg - F \sin \theta) \quad \text{decreases with} \quad \theta
\]

Puzzle  There’s a prize for the first completely correct answer to this one.

How high should \( h \) be so that it can loop the loop? Note the cross section of the track. \( h \) and \( R \) are measured from the rolling positions of the centre of the ball.
**Example.** A hydroelectric dam is 100 m tall. Assuming that the turbines and generators are 100% efficient, and neglecting friction, calculate the flow of water required to produce 10 MW of power. The output pipes have a cross section of 5 m$^2$.

Nett effect: ~ stationary water lost from top of dam, water appears with speed $v$ at bottom.

$dW = \text{work done by water}$

$= -\text{Work done on water} = -\text{energy increase of water}$

$dW = -dE = -dK - dU$

$= -(\frac{1}{2} dm v^2 - 0) - (0 - dm gh) = dm \left( gh - \frac{v^2}{2} \right)$

$P = \frac{dW}{dt} = \frac{dm}{dt} \left( gh - \frac{v^2}{2} \right)$

Problem: $v$ depends on $\frac{dm}{dt}$

Density: $\rho = \frac{\text{mass}}{\text{volume}} = \frac{m}{V}$ so $m = \rho V$

$\frac{dm}{dt} = \frac{dV}{dt} = \rho A v$

$P = \rho A v \left( gh - \frac{v^2}{2} \right)$

$v^3 - (2gh)v + \frac{2P}{\rho A} = 0$ we can solve a cubic, but it's messy. It's a one sig fig problem, so try an approximation

Neglect $v^3 \rightarrow v = \frac{P}{gh\rho A} = 2 \text{ m/s}$

and indeed we see that $v^3 \ll$ other terms. Think about this: if you were designing the generator, would you have made the kinetic energy term comparable with the potential energy or work terms?

Flow = $vA = 10 \text{ m}^3/\text{s} \rightarrow 10 \text{ tonne/s}$
Some quantitative examples:

How much work is required to accelerate a car
i) from 0 to 10 km/hr?
ii) from 100 to 110 km/hr?

(As asked, this is work done by total force; it includes negative work done by air resistance)

Work energy theorem

\[ W_{\text{total}} = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \]

i) \[ \frac{1}{2}(1000 \text{ kg}) \left( \frac{10000 \text{ m}}{3600 \text{ s}} \right)^2 - 0 = 4 \text{ kJ} \]

ii) \[ W_{\text{total}} = \ldots = 80 \text{ kJ} \]

dW = dK = \frac{1}{2} mv^2 dv

Energy density:

Small rechargeable NiCad:

600 mA.hr and 1.25 V

\[ \rightarrow (0.6 \text{ A})(3600 \text{ s})(1.25 \text{ V}) = 3 \text{ kJ} \]

(3 kJ)/(20 g) = 150 kJ/kg = 0.15 MJ/kg

Car battery:

Up to 100 Amp hours @ 12 V \rightarrow 4 \text{ MJ} < 0.5 \text{ MJ/kg}

warning: don't try to extract this quickly

Lithium ion:

100 W.hour/kg \rightarrow 0.36 \text{ MJ/kg} (some quote up to 0.9 \text{ MJ/kg})

\[ \frac{\text{MJ/litre}}{\text{MJ/kg}} \]

<table>
<thead>
<tr>
<th></th>
<th>MJ/litre</th>
<th>MJ/kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Petrol</td>
<td>29</td>
<td>45</td>
</tr>
<tr>
<td>LPG</td>
<td>22</td>
<td>34</td>
</tr>
<tr>
<td>Ethanol</td>
<td>19</td>
<td>30</td>
</tr>
<tr>
<td>Diesel</td>
<td>40</td>
<td>63</td>
</tr>
</tbody>
</table>

Speeding bullet

\[ \frac{\frac{1}{2}mv^2}{m} = \frac{1}{2} v^2 \sim \frac{1}{2} (500 \text{ m/s})^2 = 0.1 \text{ MJ/kg} \]

Example: What is the intensity of solar radiation? \( P_{\text{sun}} = 3.9 \times 10^{26} \text{ W} \). Earth is 150 million km from sun.

Intensity = \( \frac{P}{4\pi r^2} = \ldots = 1.38 \text{ kWm}^{-2} \) called 'solar constant'

above atmosphere, \_ \_ radiation
Example: $I_{\text{solar}} = 1.4 \text{ kW.m}^{-2}$ at earth, 8 light minutes from sun.

Power of sun = $I \cdot 4\pi r^2$

= $(1.4 \text{ kW.m}^{-2})4\pi (8 \text{ minutes \times \text{ speed of light}})^2$

= $4 \times 10^{26}$ Watts

$I = \frac{P}{4\pi r^2} \quad \rightarrow \quad I \propto \frac{1}{r^2}$
Energy in SHM  (Not in Mechanics syllabus but need in PHYS1231)

See Oscillations in Physclips

\[ x = A \sin \omega t \quad \quad v = \frac{dv}{dt} = A\omega \cos \omega t \]

\[ U = \frac{1}{2} kx^2 \quad K = \frac{1}{2} mv^2 \]

\[ E = U + K = \frac{1}{2} kA^2 \sin^2 \omega t + \frac{1}{2} mA^2 \omega^2 \cos^2 \omega t \]

but \[ \omega = \sqrt{\frac{k}{m}} \quad \text{so} \quad k = m\omega^2 \]

\[ E = \frac{1}{2} m\omega^2 A^2 \sin^2 \omega t + \frac{1}{2} mA^2 \omega^2 \cos^2 \omega t \]

\[ = \frac{1}{2} m\omega^2 A^2 (\sin^2 \omega t + \cos^2 \omega t) \]

\[ A\omega = v_{\text{max}} \]

\[ E = K_{\text{max}} = U_{\text{max}} \]