**PHYS 1A Work and Energy**  Joe Wolfe, UNSW

The scalar product. *dot product*

**Why?** e.g. Work: scalar, related to $\mathbf{F}$, $\mathbf{ds}$ and $\theta$.

\[
\text{dW} = |\mathbf{F}| |\mathbf{ds}| \cos \theta
\]

(also $dV = |\mathbf{E}| |\mathbf{ds}| \cos \theta$ etc)

\[\vec{a} \cdot \vec{b} \equiv ab \cos \theta \quad (= \vec{b} \cdot \vec{a})\]

Apply to unit vectors:

\[\vec{i} \cdot \vec{i} = 1 \cdot 1 \cos 0^\circ = 1 = \vec{i} \cdot \vec{j} = \vec{k} \cdot \vec{k}\]
\[\vec{i} \cdot \vec{i} = 1 \cdot 1 \cos 90^\circ = 0 = \vec{i} \cdot \vec{k} = \vec{k} \cdot \vec{i}\]

**Scalar product by components**

\[\vec{a} \cdot \vec{b} = (a_x \vec{i} + a_y \vec{j} + a_z \vec{k}) \cdot (b_x \vec{i} + b_y \vec{j} + b_z \vec{k})\]
\[= (a_x b_x) \vec{i} \cdot \vec{i} + (a_y b_y) \vec{j} \cdot \vec{j} + (a_z b_z) \vec{k} \cdot \vec{k}\]
\[+ (a_x b_y + a_y b_x) \vec{i} \cdot \vec{j} + (..) \vec{j} \cdot \vec{k} + (..) \vec{k} \cdot \vec{i}\]

\[\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z\]

**Problem.** Find the angle between

\[\vec{a} = 4 \vec{i} - 3 \vec{j} + 7 \vec{k}\]
\[\vec{b} = 2 \vec{i} + 5 \vec{j} - 3 \vec{k}\]

\[ab \cos \theta = \vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z\]

\[\cos \theta = \frac{a_x b_x + a_y b_y + a_z b_z}{\sqrt{a_x^2 + a_y^2 + a_z^2} \sqrt{b_x^2 + b_y^2 + b_z^2}}\]

\[\rightarrow \theta = 122^\circ\]

**Definition of work**

\[
\text{dW} = F \text{ ds } \cos \theta
\]

When force varies, use differential displacement $\text{ds}$

\[dW = F \text{ ds } \cos \theta \]

\[\rightarrow F \text{ component of ds } // \mathbf{F}, \text{ or}\]
\[F \cos \theta \text{ (ds)} \rightarrow \text{ ds } \text{X component of } \mathbf{F} // \text{ds}\]

\[W = \int \text{F cos } \theta \text{ ds}\]

*if F and \( \theta \) are constant, we get* \[W = FL \cos \theta\]

SI Unit: 1 Newton x 1 metre = 1 Joule
Example. How much work is done by lifting 100 kg vertically by 1.8 m very slowly?

\[ \text{Slow} \quad \therefore F_{\text{applied}} \approx mg \]
\[ W = mg \cdot d \cos \theta \]
\[ = 1.8 \text{ kJ}. \]

Not a lot, yet it is hard to do, because the force is inconveniently large.

Consider:

If the rope and pulleys are light, and if the accelerations are negligible, then

Force on LH pulley
\[ ma \approx 0 = 2T - mg \]
\[ \therefore T = mg/2 \]

If mass rises by \( D \), work done = \( mgD \).

But rope shortens on both sides of rising pulley,

if mass rises by \( D \), rope must be pulled \( 2D \),

so work done = \( T \cdot 2D = mgD \)

Example. What is the work done by gravity in a circular orbit?

\[ W = \int F \, ds \cos \theta = 0 \]

Example. \( F_{\text{grav}} \propto 1/r^2 \). How much work is done to move \( m = 1 \) tonne from earth's surface (\( r = 6500 \text{ km} \)) to \( r = \infty \)?

\[ W = \int F \, ds \cos \theta \\
= \int F \, dr \\
F = -F_{\text{grav}} = \frac{Cm}{r^2} \quad \text{more later} \]

On surface \( F/m = 9.8 \text{ ms}^{-2} \)
\[ \therefore C = (9.8 \text{ ms}^{-2})(6.5 \times 10^6 \text{ m})^2 = 4.1 \times 10^{14} \text{ m}^2\text{s}^{-2} \]
\[ W = \int_{6500 \text{ km}}^{\infty} \frac{Cm}{r^2} \, dr \\
= -Cm \left( \frac{1}{\infty} - \frac{1}{6.5 \times 10^6} \right) \\
= 6.3 \times 10^{10} \text{ J} = 63 \text{ GJ}. \]
Work to deform spring

No applied force \((x = 0)\)

Hooke's law: \(F = -kx\)

Work done by spring = \(\int F_{\text{spring}} \, dx\)

\[= \int -kx \, dx = -\frac{1}{2}kx^2\]

Work done on spring = \(\int F_{\text{applied}} \, dx\)

\[= \int kx \, dx = +\frac{1}{2}kx^2 \quad (= \text{work stored in spring})\]

The work-energy theorem

(Total) force \(F\) acts on mass \(m\) in \(x\) direction.

Work done by \(F\) = \(\int_{i}^{f} F \, dx\) \quad \text{(use } F = ma\text{)}

\[= \int_{i}^{f} m \frac{dv}{dt} \, dx = \int_{i}^{f} m \frac{dx}{dt} \, dv\]

\[= \int_{i}^{f} mv \, dv = \left[ \frac{1}{2}mv^2 \right]_{i}^{f}\]

Work done by \(F\) = \(\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 \equiv \Delta K\)

Define \textbf{kinetic energy} \(K \equiv \frac{1}{2}mv^2\)

Increase in kinetic energy of body = work done by \textbf{total} force acting on it.

\[
\text{This is a theorem, ie a tautology because it is only true by definition of KE and by Newton 2.} \\
\therefore \text{restatement of Newton 2 in terms of energy. Not a new law}
\]
Power. is the rate of doing work

Average power \( \bar{P} = \frac{W}{\Delta t} \)

Instantaneous power \( P = \frac{dW}{dt} \)

SI unit: 1 Joule per second \( \equiv \) 1 Watt (1 W)

**Example**  Jill (m = 60 kg) climbs the stairs in Matthews Bldg and rises 50 m in 1 minute. How much work does she do against gravity? What is her average output power? (neglect accelerations)

\[
W = \int F_y \, ds = \int F_y \, dy
\]

\( F_y = mg \)

\[
W = mg \int dy = mg \Delta y = 29 \text{ kJ}
\]

(cf \( K = \frac{1}{2}mv^2 \approx 20 - 40 J \))

\[
\bar{P} = \frac{W}{\Delta t} = \frac{mg \Delta y}{\Delta t} = 490 \text{ W}
\]

(humans can produce 100s of W, car engines several tens of kW)

(1 horsepower \( \approx 550 \text{ ft.lb.s}^{-1} = 0.76 \text{ kW} \))

**Potential energy.**

e.g. Compress spring, do W on it, but get no K. Yet can get energy out: spring can expand and give K to a mass. \( \rightarrow \) Idea of stored energy.

e.g. Gravity: lift object (slowly), do work but get no K. Yet object can fall back down and get K.

Recall \( W_{\text{against grav}} = mg \Delta y \) i.e. \( W = W(y) \)

**But:** Slide mass slowly along a surface. Do work against friction, but can't recover this energy mechanically. Not all forces "store" energy

**Conservative and non-conservative forces**

(same examples)

\[
W_{\text{against grav}} = -\int_i^f F_g \, dr \cos \theta
\]

\[= -\int_i^f F_g \, dz \]

\[= mg \int_i^f dz \]

\[= mg (z_f - z_i) \] \( \text{in uniform field} \)

W is uniquely defined at all \( \mathbf{r} \), i.e. \( W = W(\mathbf{r}) \)

If \( z_f - z_i \) are the same, \( W = 0 \).

\( \therefore \) Work done against gravity round a closed path = 0

Gravity is a conservative force
Compare spring

\[ W_{\text{against spring}} = -\int F_{\text{spring}} \, dx = -\int -kx \, dx \]

\[ = \frac{1}{2} k(x_f^2 - x_i^2) \]

W is uniquely defined at all x, i.e. \( W = W(x) \)
\( x_f = x_i \Rightarrow W = 0. \)

\( \therefore \) Work done round a closed path = 0

Spring force is a **conservative force**

*so it has stored or potential energy: symbol \( U \).*

**with friction**

\[ dW_{\text{against fric}} = -F_f \, ds \cos \theta \]

but \( F_f \) always has a component **opposite \( ds \)**

\( \therefore \) \( dW \) always \( \geq 0. \)

\( \therefore \) cannot be zero round closed path, \( \therefore W \neq W(\mathbf{r}) \)

\( \therefore \) friction is a **non-conservative force**

*Note that direction of friction (dissipative force) is always against motion.*

**Potential energy**

For a **conservative** force \( \mathbf{F} \) (i.e. one where work done against it, \( W = W(\mathbf{r}) \)) we can define potential energy \( U \) by \( \Delta U = \frac{dW}{dt} \).

\[ \Delta U = -\int F \, dr \cos \theta \]

Same example: **spring**

\[ \Delta U_{\text{spring}} = -\int F_{\text{spring}} \, dx \]

\[ = \frac{1}{2} k(x_f^2 - x_i^2) \]

**Choice of zero for \( U \) is arbitrary.**

Here \( U = 0 \) at \( x = 0 \) is obvious, so

\[ U_{\text{spring}} = \frac{1}{2} kx^2 \]
From energy to force:
\[ U = - \int F \, ds \] where \( ds \) is in the direction \( \parallel F \)

\[ F = - \frac{dU}{ds} \]

In fact
\[ F_x = - \frac{dU}{dx}, \quad F_y = - \frac{dU}{dy}, \quad F_z = - \frac{dU}{dz} \]

Spring: \( U_{\text{spring}} = \frac{1}{2} kx^2 \) \( \therefore F_{\text{spring}} = -kx \)

Gravity: \( U_g = mgz \) \( \therefore F_g = -\frac{dU}{dz} = -mg \)

Energy of interaction:

![Diagram of energy interaction](image)

Conservation of mechanical energy

Recall: Increase in \( K \) of body = work done by total force acting on it. \((\text{restatement of Newton 2})\)

But, if all forces are conservative, work done by these forces = \(-\Delta U\) \((\text{definition of } U)\)

\( \therefore \) if only conservative forces act, \( \Delta K = -\Delta U \)

We define mechanical energy
\[ E \equiv K + U \]

so, if only conservative forces act, \( \Delta E = 0 \).

We can make this stronger.

Work done by non-conservative forces

Define internal energy \( U_{\text{int}} \) where
\[ \Delta U_{\text{int}} = -\text{Work done by n-c forces} \]
\[ (= + \text{Work done against n-c forces}) \]

Recall def\( n \) of \( K \): \( \Delta K = \text{work done by } \Sigma \text{ force} \)

\( \therefore \quad \Delta K = -\Delta U - \Delta U_{\text{int}} \)

\( \therefore \quad \Delta K + \Delta U + \Delta U_{\text{int}} = 0 \)

If n-c forces do no work, then \( \Delta U_{\text{int}} = 0 \), so:

**If non-conservative forces do no work,**
\[ \Delta E \equiv \Delta K + \Delta U = 0 \]

or: \textbf{mechanical energy } \( E \) \textbf{ is conserved}

Equivalent to Newton 2, but useful for many mechanics problems where integration is difficult.

State the principle carefully! \textbf{Never, ever write}: "kinetic energy = potential energy"
Classic problem. Child pushes off with $v_i$. How fast is the s/he going at the bottom of the slide? Neglect friction (a non-conservative force).

![Diagram of a child on a slide](image)

i) By Newton 2 directly:

$$v = \int_{\text{top}}^{\text{bottom}} a \, dt = \int_{\text{top}}^{\text{bottom}} \frac{F}{m} \, dt = \int_{\text{top}}^{\text{bottom}} g \cos \theta \, dt = ...$$

ii) Using work energy theorem (Newton 2 indirectly):

Non-conservative forces do no work, hence mechanical energy is conserved, i.e.

$$\Delta E = \Delta K + \Delta U = 0 \quad \text{or} \quad E_f = E_i$$

$$K_f - K_i + U_f - U_i = 0 \quad \text{or} \quad K_f + U_f = K_i + U_i$$

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 + mg_y - mg_y_i = 0$$

rearrange $\rightarrow v_f = \sqrt{v_i^2 + 2g(y_i - y_f)}$

**Conservation of energy**

observation: for many forces, $W = W(r)$, useful to define $U = U(r)$.

observation: for all systems yet studied, $U_{\text{int}}$ is a state function, i.e. $U_{\text{int}} = U_{\text{int}}(\text{measured variables})$

Hence idea of internal energy. E.g.:

*Friction, $-U_{\text{int}}$ = heat produced when work is done against friction.*

*Air resistance $-U_{\text{int}}$ is sound and heat.*

*Combustion engines and animals: $+U_{\text{int}}$ comes from chemical energy*

$$\Delta K + \Delta U + \Delta U_{\text{int}} = 0$$

is statement of Newton 2 plus definitions of $K$, $U$, $U_{\text{int}}$.

The statement that $\Delta U_{\text{int}}$ is a state function is the first law of thermodynamics. It is a law, i.e falsifiable. More on this in Heat.

**Example.** Freda ($m = 60 \text{ kg}$) rides pogo stick ($m \ll 60 \text{ kg}$) with spring constant $k = 100 \text{ kN.m}^{-1}$. Neglecting friction, how far does spring compress if jumps are 50 cm high?

![Diagram of a pogo stick](image)

Non-conservative forces do no work, hence mechanical energy is conserved, i.e.

$$E_{\text{bottom}} = E_{\text{top}}$$

$$K_b + U_b = K_t + U_t \quad \left( U = U_{\text{grav}} + U_{\text{spring}} \right)$$

$$\frac{1}{2}mv_{\text{horiz}}^2 + (mg y_b + \frac{1}{2}kx_b^2) \equiv \frac{1}{2}mv_{\text{horiz}}^2 + (mg y_t + \frac{1}{2}kx_t^2)$$

$$mg(y_t - y_b) \equiv \frac{1}{2}kx_b^2$$

$\therefore x_b \equiv \sqrt{\frac{2mg(y_t - y_b)}{k}} \equiv 80 \text{ mm.}$
Example. Slide starts at height $h_1$. Later there is a hump with height $h_2$ and (vertical) radius $r$. What is the minimum value of $h_2 - h_1$ if slider is to become airborne? Neglect friction, air resistance.

Over hump, $a_c = \frac{v^2}{r}$ (down). Airborne if $g < a_c$, i.e. if $v_2^2 > gr$.

No non-conservative forces act so

$$E_2 = E_1$$

$$U_2 + K_2 = U_1 + K_1$$

$$mgy_2 + \frac{1}{2}mv_2^2 = mgy_1 + \frac{1}{2}mv_1^2$$

$$\frac{1}{2}mv_2^2 = mg(y_1 - y_2)$$

$$(y_1 - y_2) = \frac{v_2^2}{2g} > \frac{gr}{2g} = \frac{r}{2}$$

Example  Bicycle and rider (80 kg) travelling at 20 m.s$^{-1}$ stop without skidding, $\mu_s = 1.1$. What is minimum stopping distance? How much work done by friction between tire and road? Between brake pad and rim? Wheel rim is ~300 g with specific heat ~ 1 kj.kg$^{-1}$, how hot does it get?

**friction $\rightarrow$ deceleration $\rightarrow$ stopping distance**

$$|a| = \frac{F_f}{m} \leq \frac{\mu_s N}{m} = \mu_s g$$

$$|a| \leq \mu_s g$$

$$v_f^2 - v_i^2 = 2as \quad \Rightarrow \quad s = \frac{v_f^2 - v_i^2}{2a}$$

$$s \geq \left| \frac{-v_i^2}{2\mu_s g} \right| = 19 \text{ m}$$

Work done by friction between tire and road? No skidding, $\therefore$ no relative motion, $\therefore$ $W = 0$.
Between pad and rim? Here $\exists$ relative motion. All K of bike & rider $\rightarrow$ heat in rim and pad

$$W = \Delta K = K_f - K_i = -16 \text{ kJ}$$

$$Q = mC\Delta T \quad \therefore \Delta T \sim 50 \text{ °C}$$
Example Which way is it easier to drag an object?

Suppose we move at steady speed, \( a = 0 \). Which requires less \( F \)? Which requires less work?

mechanical equilibrium → horizontal \( F \cos \theta = F_f \)

sliding → \( F_f = \mu_k N \)

eliminate \( N \) → \( F \cos \theta = \mu_k (mg - F \sin \theta) \)

\[ F = \frac{\mu_k mg}{\cos \theta + \mu_k \sin \theta} \]

when \( \theta = 0 \), \( F' = \mu_k mg \)

\( F < F' \) if \( \cos \theta + \mu_k \sin \theta > 1 \), i.e. if \( \mu_k \) large & \( \theta \) small

Work done = \( Fs \cos \theta = F_f s \)

= \( \mu_k N s = \mu_k s (mg - F \sin \theta) \) decreases with \( \theta \)

Question

How high should \( h \) be so that it can loop the loop?
**Example.** A hydroelectric dam is 100 m tall. Assuming that the turbines and generators are 100% efficient, and neglecting friction, calculate the flow of water required to produce 10 MW of power. The output pipes have a cross section of 5 m².

Nett effect: ~ stationary water lost from *top* of dam, water appears with speed \( v \) at bottom.

Let flow be \( \frac{dm}{dt} \).

\[
\begin{align*}
dW &= \text{work done by water} \\
\quad &= - \text{Work done on water} = - \text{energy increase of water} \\
\quad &= - \left( \frac{1}{2} \frac{dm}{dt} v^2 - 0 \right) - \left( 0 - \frac{dm}{dt} gh \right) = \frac{dm}{dt} \left( gh - \frac{v^2}{2} \right)
\end{align*}
\]

\[
\begin{align*}
\mathcal{P} &= \frac{dW}{dt} = \frac{dm}{dt} \left( gh - \frac{v^2}{2} \right)
\end{align*}
\]

Problem: \( v \) depends on \( \frac{dm}{dt} \)

\[
\begin{align*}
v \cdot dt = A \cdot \frac{dV}{dt} = A \cdot \frac{dm}{dt}
\end{align*}
\]

Density: \( \rho \equiv \frac{\text{mass}}{\text{volume}} = \frac{m}{V} \) so \( m = \rho V \)

\[
\begin{align*}
\frac{dm}{dt} &= \rho \frac{dV}{dt} = \rho Av \\
\mathcal{P} &= \rho Av \left( gh - \frac{v^2}{2} \right)
\end{align*}
\]

\[
\begin{align*}
v^3 - (2gh)v + \frac{2\mathcal{P}}{\rho A} &= 0 \\
\text{can solve cubic, but messy}
\end{align*}
\]

Neglect \( v^3 \) \( \Rightarrow \) \( v = \frac{\mathcal{P}}{gh\rho A} = 2 \text{ m/s} \)

*and indeed we see that* \( v^3 \ll \text{other terms} *