Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object.
- For an arbitrary axis, the parallel-axis theorem often simplifies calculations.
- The theorem states $I = I_{CM} + MD^2$
  - $I$ is about any axis parallel to the axis through the centre of mass of the object.
  - $I_{CM}$ is about the axis through the centre of mass.
  - $D$ is the distance from the centre of mass axis to the arbitrary axis.
The element \( dm \) is distance \( r = (x^2 + y^2)^{\frac{1}{2}} \) from the axis

\[
\therefore I = \int r^2 dm = \int (x^2 + y^2) dm
\]

Express the equation above in terms of the CoM coordinates:

\[
\Rightarrow I = \int [(x' + x_{cm})^2 + (y' + y_{cm})^2] dm
\]

\[
= \int [(x')^2 + 2x'x_{cm} + (x_{cm})^2 +
[(y')^2 + 2y'y_{cm} + (y_{cm})^2] dm
\]

\[
= \int [(x')^2 + (y')^2] dm + 2x_{cm} \int x'dm +
2y_{cm} \int y'dm + (x_{cm}^2 + y_{cm}^2) \int dm
\]

= 0 because the coordinates are relative to the CoM

\[
I = I_{cm} + MD^2
\]
\[ I = \int [(x')^2 + (y')^2] \, dm + 2x_{cm} \int x' \, dm + 2y_{cm} \int y' \, dm + (x_{cm}^2 + y_{cm}^2) \int dm \]

**Question:** Why are these 2 integrals zero?

**Answer:** We are integrating \( x' \) and \( y' \) and \( x' \) and \( y' \) are measured from the CM.

Consider one of the integrals, e.g. \( y' \). The contribution to the quantity inside the integral from distances with \( y' > 0 \) will be exactly balanced by distances with \( y' < 0 \). In other words, the integral sums to zero. It's the same for \( x' \).
Moment of Inertia for a Rod Rotating Around One End

- The moment of inertia of the rod about its centre is
  \[ I_{CM} = \frac{1}{12} ML^2 \]
- \( D \) is \( \frac{1}{2} L \)
- Therefore,
  \[ I = I_{CM} + MD^2 \]
  \[ I = \frac{1}{12} ML^2 + M \left(\frac{L}{2}\right)^2 = \frac{1}{3} ML^2 \]
Torque

- Torque, $\tau$, is the tendency of a force to rotate an object about some axis
  - Torque is a vector, but we will deal with its magnitude first
  - $\tau = r F \sin \phi = F d$
    - $F$ is the force
    - $\phi$ is the angle the force makes with the horizontal
    - $d$ is the *moment arm* (or lever arm) of the force

- The moment arm, $d$, is the *perpendicular* distance from the axis of rotation to a line drawn along the direction of the force
  - $d = r \sin \Phi$
- The horizontal component of the force ($F \cos \phi$) has no tendency to produce a rotation
- Torque will have direction
  - If the turning tendency of the force is counterclockwise, the torque will be positive
  - If the turning tendency is clockwise, the torque will be negative

**Net Torque**
- The force $\vec{F}_1$ will tend to cause a counterclockwise rotation about $O$
- The force $\vec{F}_2$ will tend to cause a clockwise rotation about $O$
- $\Sigma \tau = \tau_1 + \tau_2 = F_1d_1 - F_2d_2$
Torque vs. Force

- Forces can cause a change in translational motion
  - Described by Newton’s Second Law
- Forces can cause a change in rotational motion
  - The effectiveness of this change depends on the force and the moment arm
  - The change in rotational motion depends on the torque

- The SI units of torque are N·m
  - Although torque is a force multiplied by a distance, it is very different from work and energy
  - The units for torque are reported in N·m and not changed to Joules
Torque as a vector

We saw before that torque is defined as the product of the applied force and the distance from point of application to point of rotation. Force and distance are both vectors. Hence, we need a new way of combining vectors, the \textit{cross-product} or \textit{vector product}:

\[ \vec{A} \times \vec{B} = \vec{C} = AB \sin \theta \hat{n} \]

So the magnitude is \[ |\vec{A} \times \vec{B}| = AB \sin \theta \]

\( \theta \) is the angle between \( \vec{A} \) and \( \vec{B} \)

The direction of the resultant vector is given by the right hand rule
To get the direction of the cross-product:

1. Using right hand, point fingers in direction of $\vec{A}$
2. Curl fingers towards $\vec{B}$
3. Thumb now points along $\vec{C}$

Notes:

i) Order of cross-product is important!

ii) If vectors are parallel, cross-product is zero

iii) If vectors are at right-angles, cross-product is maximum
Torque and Angular Acceleration

- Consider a particle of mass $m$ rotating in a circle of radius $r$ under the influence of tangential force $\vec{F}_t$
- The tangential force provides a tangential acceleration:
  - $F_t = ma_t$
- The radial force, $\vec{F}_r$, causes the particle to move in a circular path
The magnitude of the torque produced by $\sum \vec{F}_t$ around the center of the circle is

$\Sigma \tau = \Sigma F_t \cdot r = (ma_t) \cdot r$

The tangential acceleration is related to the angular acceleration

$\Sigma \tau = (ma_t)r = (mr\alpha)r = (mr^2)\alpha$

Since $mr^2$ is the moment of inertia of the particle,

$\Sigma \tau = I\alpha$

The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia.
Consider the object consists of an infinite number of mass elements $dm$ of infinitesimal size

Each mass element rotates in a circle about the origin, $O$

Each mass element has a tangential acceleration

- From Newton’s Second Law
  - $dF_t = (dm) a_t$

  The torque associated with the force and using the angular acceleration gives
  - $d\tau = r dF_t = a_t r \, dm = \alpha r^2 \, dm$

Finding the net torque
  - $\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm$

  This becomes $\sum \tau = I\alpha$
Falling Smokestack Example

- When a tall smokestack falls over, it often breaks somewhere along its length before it hits the ground.
- Each higher portion of the smokestack has a larger tangential acceleration than the points below it.
- The shear force due to the tangential acceleration is greater than what the smokestack can withstand.
- The smokestack breaks.
Calculate the **angular acceleration** of the wheel, the **linear acceleration** of the object and the tension in the cord.

Torque on wheel is $TR$, i.e. $\sum \tau = I \alpha \Rightarrow TR = \alpha$  

$$\alpha = TR/I$$

Mass moving in straight line so apply Newton’s 2$^{nd}$,  

$\sum F = mg - T = ma \Rightarrow a = (mg - T)/m$

Three unknowns ($\alpha$, $a$ & $T$) but only two equations - need one more

Object and wheel connected by cord which does not slip, so linear acceleration is same as tangential  

$a = R \alpha = TR^2/I = (mg - T)/m$

$$\Rightarrow T = mg/[1 + (mR^2/I)]$$

$$\Rightarrow a = g/[1 + (I/mR^2)]$$

$$\Rightarrow \alpha = a/R = g/[R + (I/mR)]$$
Newton’s Law of Universal Gravitation

- Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the squared distance between them ("inverse square law"):

\[ F_g = G \frac{m_1 m_2}{r^2} \]

- \( G \) is the \textit{universal gravitational constant} and equals 6.673 \times 10^{-11} \text{ N m}^2 / \text{kg}^2

- Current We have no theoretical explanation for its value!

- Gravity is \(10^{32}\) times weaker than the electromagnetic force!

- Current known precision on the relative uncertainty is \( \frac{\Delta G}{G} \approx 10^{-6} \) (Rosi et al, Nature 510, 518–521 (2014) )

- In vector form \( \vec{F}_{12} = -G \frac{m_1 m_2}{r^2} \hat{r}_{12} \)
More About Forces

- $\vec{F}_{12} = -\vec{F}_{21}$
  - The forces form a Newton’s Third Law action-reaction pair
- Gravitation is a field force that always exists between two particles, regardless of the medium between them
- The force decreases rapidly as distance increases (due to the inverse square law)

$F_{12}$ is the force exerted by particle 1 on particle 2
The negative sign in the vector form of the equation indicates that particle 2 is attracted toward particle 1

$F_{21}$ is the force exerted by particle 2 on particle 1
Gravitational Force Due to a Distribution of Mass

- The gravitational force exerted by a finite-size, spherically symmetric mass distribution on a particle outside the distribution is the same as if the entire mass of the distribution were concentrated at the center.
- In other words - on or above the Earth’s surface, all of its mass appears to emanate from a point at its centre (last part also true inside spherical distribution).

The force exerted by the Earth on a particle of mass $m$ near the surface of the Earth is

$$F_g = G \frac{M_E m}{R_E^2}$$
If you are here, the gravitation attraction due to matter “above” is ...