CA8VSA Ø2.DOC

SECOND HARMONIC GENERATION IN VIOLINS

Summary

Second harmonic generation in violins is not augmented significantly by any change in the string tension during the periodic traverse of the kink around its parabolic circuit. A calculation suggests the size of such an effect is insignificant, and therefore supports a 1937 experimental finding.

Introduction

This paper is written in response to statements made at a V.S.A. Makers Forum, Campbell(1992), that the second harmonic or "octave effect which for the modern fiddle is 50 percent of the total volume of sound" and gives the "loudest component of the output". This idea arises from the fact that the tension in a vibrating string cycles at twice the frequency of the fundamental and the effect of this should be in addition to that of the kink as it deflects the bridge. An argument was put forward that the fore and aft bending of the bridge as distinct from the rocking motion caused an additional deflection of the top plate. The purpose of this paper is to explore that idea and to show that the octave effect is too small to be audible. First, some general features of string behaviour will be outlined, principally those associated with in-plane and out-of-plane motion, and second, the order of magnitude of deflections to be expected at the bridge and top plate.

Minnaert and Vlam (1937) studied the motions of the violin bridge. They found in-plane rocking and translational motions and out-of-plane flexing and torsional motions. The octave effect would involve the out-of-plane flexural motion of the bridge. Their conclusion is worth quoting.

> "The general result of our investigation is, that the bridge executes different complicated movements, as well in its own plane as perpendicularly to it. These correspond to the proper vibrations of several vibrating systems, consisting of the whole of the strings, the tailpiece, the belly or the bridge itself, in some respects strongly coupled to each other. Thanks to the cuts in the bridge, only the vibration in its plane is transmitted to the feet, and by them to the belly of the instrument."

This paper presents an argument in support of this conclusion in that if any motion was transmitted to the bridge feet it would be so small as to have no noticeable effect. A calculation of the transverse force at the string notch due to the change in string tension is made and compared with the force due to the Helmholz motion. The restraining effect of the other strings as well as the effect of resonances has not been taken into account. Before dealing with the variation in string tension, a word about the rocking or in-plane motion of the bridge might be in order to demonstrate what the string is doing. The following treatment is based on the Helmholtz description of string behaviour.

Idealised String Behaviour

This description of string behaviour is related to the Helmholtz (1877 p.384) explanation of bowed string action in which the ordinate of the kink moves back and forth from nut to bridge at constant velocity. The in-plane deflection of the bridge is governed by the force exerted at the top of it as the kink in the string approaches and immediately departs. The maximum value of this force depends on the amplitude of the string vibration (which in turn depends on the bowing position and bow velocity).

Figure 1 shows diagrammatically the various parameters associated with the string motion. The initial deflection and force on the bridge caused by the string as the bow is applied have been ignored. During the "steady state" motion of the string the kink traces out the segments of two parabolas in its excursion round the circuit between bridge and nut. The string may be likened to a whip cracking when it reaches either the bridge or the nut.

The parabolic envelope that the kink traces out in its journey from nut to bridge can be expressed in the form (see figure 2 and the appendix):

$$y = 4Y(1 - x/L)x/L \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Y has the value $v_b/(8\beta f)$. The terms used are defined and the equations are developed in the appendix. The maximum increase in string length, dL_{max}, for the bowing conditions specified, when the kink is at L/2 is:

$$dL_{max} = (1/2L)(v_b/(4\beta f))^2 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

and the corresponding increase in tension, dT_{max}, is given by:

$$dT_{max} = (S_L/2L)(v_b/(4\beta f))^2 \quad \ldots\ldots\ldots\ldots\ldots(3)$$

where S_L is the longitudinal string stiffness which is defined in the next paragraph. These equations are used in the numerical examples below.

When the string is deflected by the bow its length is increased, raising the string tension. This increase in string tension will depend on the elastic modulus of the string material. For gut and nylon cored strings this will be the unrelaxed modulus since they are "standard anelastic solids". For metal cored strings Young's

Modulus can be used without as much error as would be incurred if used with the other materials. One can express the increase in string tension in terms of string stiffness, $S_L = E_u a/L$, where E_u is the unrelaxed modulus (or Young's Modulus), a is the string cross sectional area, and L is the string length from bridge to nut. The increase in string tension is $dT = S_L dL$ where dL is the increase in string length produced by the deflection. This increase in string tension will increase the down bearing of the bridge on the top plate by $dD = 2dT \cos(\theta/2)$ where θ is the obtuse angle made by the string as it passes over the bridge. It is assumed that the line of action of the down-bearing halves this angle and passes through the bridge. The extent of the deflection of the top plate will depend on its effective stiffness.

Determination of String Elastic Moduli and Stiffness

The relaxed and unrelaxed moduli were determined for a gut D-string (dia. 0.82 mm). A value for the relaxed modulus was found to be 1.68×10^9 N/m^2. This was less than the value quoted by Bell and Firth (1986) who did not distinguish between the two moduli for a standard anelastic material. The unrelaxed modulus was determined starting with a static tension of 3.5 kg wt, a value near the tuning pitch of gut strings, and was found to be 4.2×10^9 N/m^2. This value was used in subsequent calculations.

The stiffness of the two parts of the string using the value determined for the unrelaxed modulus is $S_L = 6.7 \times 10^3$ N/m for the 330 mm length between the nut and the bridge, and $S_1 = 40.3 \times 10^3$ N/m for the 55 mm length between the bridge and the tailpiece.

The calculation of interest is the amount of flexing expected at the top of the bridge.

Rocking (in-plane) Motion of the Bridge

It is of interest to calculate the effect of the increase in tension on the magnitude of the rocking motion responsible for the sound production in the violin for comparison with the flexural motion thought to contribute to the "octave effect". We therefore begin by estimating the in-plane transverse force due to the kink and that due to the increase in tension. From these the deflections at the bass foot of the bridge can be found. This will depend on two factors; the transverse force exerted at the top of the bridge by the vibrating string and the resistance exerted at the bridge feet. The force exerted by the string will be proportional to the bowing velocity, v_b (in this paper taken at 1 m/s), and inversely proportional to the relative distance of the bow from the bridge, β (here taken at 0.1). The resistance present at the bridge feet will depend on the effective stiffness of the body. If we assume the bridge rocks about the treble bridge foot we need only consider motion at the bass foot.

Following Cremer (1984) equation 2.2, namely $F_y = T_x(dy/dx)$, where F_y is the transverse force, T_x is the string tension (Cremer uses F_x) and dy/dx is the slope of the parabola at the bridge, we can obtain an estimate of the transverse force for the bowing parameters used here. Using equation (1) taking T_x = 25 N, (a value calculated for the gut string used here, see below) we can determine Y to a first approximation since the additional tension induced by bowing is less than one tenth of this. This can be used to calculate y for a particular x on this parabola. Thus from the slope at x = 0, dy/dx = 4Y/L gives the transverse force F_y = 1.26 N. This can be translated to the bass foot of the bridge and becomes 1.59 N on applying the bridge lever ratio. The corresponding deflection at the bass foot is 2.7×10^{-6} m (2.7 μm).

We can compare these estimates with two from the literature, the first for the transverse force exerted by the string at the top of the bridge. Using Cremer (1984) equation 3.22 (corrected, see note with reference) viz: $F(0) = (T_x m')^{1/2} v_b/\beta$ where F(0) is the peak value of the transverse force on the bridge top for the string tension, T_x; m′ is the mass per unit length of the string. The gut D string used as an example in this work, allows us to calculate a value for the tension from $T_x = m'(2Lf)^2$ giving 25 N. The jump in force is twice F(0) at reversal. $(T_x m')^{1/2}$ is the string wave impedance, Z, which has been determined by a number of workers. Putting values for T_x = 25 N and m′ = 0.64×10^{-3} kg/m with the bowing conditions given in the appendix (v_b/β = 10) we get a value of 0.9 N for the transverse force F(0). This transverse force is at the top of the bridge in the bowing direction. When resolved in the "rocking" direction (see figure 3) the lever action of the bridge increases it to 1.1 N.

Another estimate can be made using the results obtained by Meinel (1937). He found an amplitude of 28 μm at the bass foot of the bridge for a body mode at 488 Hz. The effective stiffness for the B1+ body mode at 540 Hz in another violin, which appears to be equivalent to the above mode in Meinel's paper, has been determined by a method outlined by Schelleng (1963) to be 0.8×10^6 N/m (unpublished work). Meinel's top plate thickness was about 2.5 mm compared with about 3 mm in this work. If we take an effective stiffness for this body mode in Meinel's paper of one tenth of that above we get a transverse force of 1.80 N at the bridge top.

The contribution made by the increase in string tension to the transverse force can be calculated in the following way. The increase in tension resolved in the plane of the bridge is given by:

$$dT_T = dT \sin \phi \qquad \text{(see figure 2)}$$

Of interest is the relative magnitude of dT_T, the transverse contribution of the increase in string tension, compared with the transverse force, $F_y = T_x\, dy/dx$, due to the motion of the Helmholtz kink. This can be expressed as a ratio as follows, taking note that $dy/dx = \tan \phi \cong \sin \phi \cong \phi = 4Y/L$.

$$dT_T/F_y = (dT_{max} \sin \phi)/T_x\, dy/dx$$

$$= dT_{max}/T_x \quad \text{.................................(4)}$$

Putting in the values for the parameters given earlier, we find a ratio of 0.028. Compared with the transverse force due to the Helmholtz kink, the increase in transverse force due to an increase in string tension is negligible. The maximum of the latter is not in phase with the maximum of the Helmholtz transverse force but leads (or lags) by a quarter period. (figure 1).

The down bearing, D, due to the static string tension, T_x, can be found from the equation:

$$D = 2T_x \cos \theta/2 \quad \text{.........................(5)}$$

where θ (taken as 160°) is the angle made by the string as it passes over the bridge, and assuming the bisector lies within the substance of the bridge. The extra down bearing due to the increase in string tension is obtained using equation (3) giving:

$$dD = S_L/L(v_b/(4\beta f))^2 \cos \theta/2 \quad \text{............(6)}$$

Putting in the appropriate numbers we get for dD the value of 0.245 N.

Flexural (out-of-plane) Bridge Motion

In addition to the above effect, the increase in string tension also extends the length of the string, l, between the bridge and the tailpiece, nominally 55 mm. The stiffness of this string segment $S_T = E_u a/l$ and the extension, dl, will be $dl = dT/S_T$. This will allow the bridge top to flex towards the fingerboard. As the kink moves around the circuit, the extension of this segment follows the variation in string tension, dT, in a manner depicted in figure 1. The maximum value of this change in string tension lags that of the transverse bridge force by a quarter period. The string length, L, changes in a non linear way as the kink travels round the circuit. The resulting change in string tension with time assumes a parabolic form which is concave downwards, as shown in figure 1.

The Size of the "Octave Effect"

If we take some representative values of the string properties, an estimate can be made of the size of the effect. With the above limitations imposed and considering a "static" situation minimum values only will be expected. However their order of magnitude may be helpful in deciding the importance of the issue.

The bowing conditions used correspond to a string amplitude of 4.2 mm calculated from an appropriate expression for Y . The corresponding extension dL_{max} from (A4) is 0.105 mm (0.00032 or 0.032% of the string length). The increase in string tension we are seeking then is 0.705 N (about 0.02 or 2.0% of string tension) from $dT = S_L dL$, and can calculate the extension, dl, in the length of string behind the bridge due to dT_{max}. It turns out to be 0.0175 mm (17.5 μm).

The possibility of the flexural motion of the bridge, assumed rigid, caused by a force on the top is the lever action if one edge (here assumed to be the back edge) of the bridge feet is regarded as a fulcrum. Taking the bridge height as 34 mm and the thickness at the feet as 4 mm the deflection of 0.0175 mm at the top becomes 0.0021 mm (2.1 μm) at the front edge of the feet. How the bridge would "rock" in this flexural mode is open to speculation; whether about the back or front edge or about the line of action of the downbearing.

As a force transmission this "lever" action of the bridge using the above dimensions and assumptions, would translate a force of 0.705 N normal to the bridge top of 0.705 x 34/4 = 6.0 N at the front edge of the bridge feet.

The increase in string tension of 0.705 N if transmitted as an increase in downbearing translates to a value of 0.25 N using the equation $dD = 2dT \cos(\theta/2)$. If this is applied to the top plate at the B1+ body mode (540 Hz) of effective stiffness 0.8×10^{6} N/m, the deflection would be 0.31×10^{-6} m i.e. 0.31 μm. If the B1- body mode (470 Hz) of effective stiffness 0.6×10^{6} N/m is used, the deflection expected would be 0.25/0.6E6 = 0.42×10^{-6} m i.e. 0.42 μm. These numbers while probably invalid suggest that any "Octave effect" would be quite small.

This is the most extreme case in favour of second harmonic enhancement. As my referee points out, I have omitted the possible bending of the bridge. Minnaert and Vlam (1937) found that bending took place above the waist. The degree of bending would depend on the stiffness of the bridge.

Discussion

Typical parameters for a gut string e.g. playing open D, lead to an estimate of the amplitude of flexural motion at the top of the bridge of 0.0175 mm (17.5 μm). Translated to the bridge feet rocking about the rear edge, this amplitude becomes 0.0021 mm (2.1 μm). In terms of the increase in string tension of 0.705 N expressed as an increase in downbearing, results in a deflection of the top plate at the bass foot of the bridge of 0.00042 mm (0.42 μm).

This is to be compared with the effect of the transverse force at the top of the bridge due to the motion of the kink. For the example used here, the transverse force was calculated to be 1.26 N (using an equation due to Cremer) and which appears as a downbearing of 1.59 N at the bass foot of the bridge. Another estimate based on Cremer gives a lower value. The transverse force due to the extra tension is 0.035 N from equation (4). This is small compared to the transverse force due to the Helmholtz kink.

If these considerations are valid the effect would be too small and in any event, the conclusion of Minnaert and Vlam still stands and any out-of-plane motion of the bridge is "filtered out" by the cuts in the sides of the bridge. With respect to the opening statements that appear in the introduction regarding second harmonic enhancement, on examining a number of harmonic analyses of bowed notes including both open and stopped strings, no prominent and consistent second harmonic enhancement that would support those statements has been noticed (unpublished work). Recalling the limitation set out in the introduction that a bridge with only one string was being considered, the extra restraining effect of three additional strings behind the bridge should reduce the deflection due to the vibrating string to an even smaller amount.

J.E.McLennan

5 Joanna Close (off Bula Street)

Charlestown, N.S.W. 2290

Australia

Acknowledgement

I wish to thank Dr Rod Day for assistance in the preparation of this paper and Professor Neville Fletcher for helpful advice in dealing with many of the finer points.

Appendix A

The parabola traced out by the kink as it travels from the nut to the bridge during half its excursion (periodic cycle) is given by (see Figure 2):

$$y = 4Y(1 - x/L)x/L \quad \text{.....................(A1)}$$

$$y = Y \text{ at } x = L/2$$

This can be written as

$$y = 4Y/L^2(L - x)x \quad \text{......................(A2)}$$

$= A(L - x)x$ where $A = 4Y/L^2$. Cremer (1984) equations 3.17 and 3.18, allow us to write $Y = V/8f$ where V is the string velocity difference between sticking and slipping and f is the frequency. Substituting $V = v_b/\beta$ where v_b is the bow velocity and $\beta = x/L$ is the bow position as a fraction of the string length, we can write A in the form $v_b/(2\beta L^2 f)$. Using $c = 2Lf$, $A = v_b/(\beta Lc)$.

The increase in string tension will be determined by the increase in length and the elastic modulus of the string material. The increase in length is the sum of the two string segments on either side of the kink less the original string length. This assumes the initial static deflection of the string by the bow can be neglected.

For small amplitudes we can use the binomial expansion of

Pythagoras' Theorem to find the length of the segments.

$$l_1 = x + y^2/2x; \qquad l_2 = (L - x) + y^2/(2(L - x))$$

Then $$l_1 + l_2 = L[1 + y^2/(2x(L - x))] \quad \text{.............. (A3)}$$

Thus the increase in length, $dL = Ly^2/(2x(L - x))$. The maximum increase occurrs at $x = L/2$ and is $dL_{max} = 2Y^2/L$. Substituting for $Y = L^2A/4$ where $A = v_b/(2\beta L^2 f)$ we get:

$$dL_{max} = (2L)^{-1}[v_b/(4\beta f)]^2 \text{.................. (A4)}$$

The increase in tension corresponding to this increase in length can be determined from the stiffness of the active part of the string, giving for dT_{max}:

$$dT_{max} = (S_L/2L)[v_b/(4\beta f)]^2 \text{ .... (A5)}$$

This maximum increase in tension will be felt at the bridge and directed at an angle, ϕ, which is the angle made by the string segment with the undeflected position of the string in this case. The direction of this tension of interest in the present argument is that normal to the plane of the bridge (see Figure 2), namely

$$dT_N = dT \cos\phi$$

We can if we wish calculate ϕ for the position of the kink at $x = L/2$ and therefore dT_{max}, however $\cos\phi$ can be taken as equal to one:

$$\text{Cos}\,\phi = x/l_1 = 1/(1 + y^2/2x^2)$$

From (A2) $$y^2/x^2 = A^2(L - x)^2$$

$$\cos\phi = 1/(1 + A^2/2(L - x)^2) \text{ .... (A6)}$$

$$= 1/[1 + 1/2(v_b/(2\beta L^2 f))^2(L^2 - 2xL + x^2)]$$

At $x = L/2$ $$\cos\phi = 1/[1 + (1/8)(v_b/2\beta Lf)^2]$$

$$= 1 - (1/8)(v_b/(2\beta Lf))^2$$

The force increase dT_N will be felt by the string length between the bridge and the tailpiece and will result in an extension dl that will be equal to the maximum flexural displacement of the top edge of the bridge. The displacement will be from a starting position and back. The extent of this movement can be calculated as:

$$dl = dT_{max}/S_1 \cos \phi$$

where S_1 is the stiffness of that part of the string between the bridge and the tailpiece. Then

$$dl = (dT_{max}/S_1)[1/(1 + 1/8(v_b/(2\beta Lf))^2)] \quad .....(A7)$$

Substituting for dT_{max} above

$$dl = (S_L/LS_1)[(v_b/(4\beta f))^2][1/[1 + 1/8(v_b/(2\beta Lf))^2]]$$

Using values for the gut string quoted in this paper, namely $S_L = 6.7\times10^3$ N/m, $S_1 = 40.3\times10^3$ N/m and $v_b = 1$ m/s, $\beta = 0.1$, $L = 0.33$ m, $f = 300$ Hz for D_4, we get $dl_{max} = 35.0\times10^{-6}$ m, i.e. 35 μm.

From $\cos \phi$ (at $x = 0$) $= 1/[1 + (1/2)(v_b/(2\beta Lf))^2]$ we can calculate for chosen bowing conditions, the angle ϕ that the string makes with the normal to the plane of the bridge (ignoring the initial string offset). For the conditions above for the gut D string, $\phi = 2.89°$. The value of ϕ calculated when $dT = dT_{max}$, at $x = L/2$, is 1.45°. This angle is dependent on the value of β and becomes larger as β decreases (or v_b/β increases).

Appendix B

In the event of more than one Helmholtz kink, the increase in length would be greater than that determined above. In the limit with an infinite number of kinks, the string length would be that of the parabolic arc. The above limit is an impossible physical situation although more than one kink is possible.

The length of the parabolic arc can be found from:

$$S = \int_0^L [1 + (dy/dx)^2]^{1/2}\, dx \quad \ldots\ldots\ldots\ldots\ldots (B1)$$

where $y = A(L - x)x$

Solving the expression (B1) by putting it in the standard form containing $(x^2 + a^2)^{1/2}$ and substituting $A = v_b/(\beta Lc)$ we arrive at:

$$S = (L/2)[(AL)^2 + 1]^{1/2}$$

$$+ (1/4A)\log[2(AL)^2 + 1 + 2AL[(AL)^2 + 1]^{1/2}] \quad \ldots (B2)$$

$$= (L/2\beta c)[v_b^2 + (\beta c)^2]^{1/2}$$

$$+ (\beta Lc/4v_b)\log[2v_b^2/(\beta c)^2 + 1 + 2v_b/(\beta c)^2[v_b^2 + (\beta c)^2]^{1/2}] \quad ..(B3)$$

The increase in length over the undeflected string can be found by subracting L. The wave velocity c can be replaced by 2Lf as noted above. For the bowing conditions used in this paper, the length of the parabolic arc is 0.035 mm longer than the maximum length of the string with one Helmholz kink. This is 0.01 % of the original string length.

References

Bell A.J. and Firth I.M., "The physical properties of gut musical instrument strings", Acustica, 60, 87-89, 1986.

Campbell E.C.(1992) Makers Forum, Jnl VSA, XII(2), 139-155.

Cremer L.(1984), "The Physics of the Violin", MIT Press. (There appears to be a typographical error with a "1" where an "l" was intended in equation 3.22 with the consequent dimension error).

Helmholtz H.L.M.(1877), "On the Sensations of Tone", Dover Publications Inc., New York 1954.

Meinel H. (1937), "Concerning the relationship between thickness of wood, mode of vibration, body amplitude and timbre of a violin body", Elekt. Nachr.-Tech. 14(4), 119-134, (reprinted in Musical Acoustics, 2, 56, 1976).

Minnaert M. and Vlam C.C.(1937), "The vibrations of the violin bridge", Physica, 4(5), 361-372. (reprinted in Musical Acoustics, 1, 281, 1976)

Schelleng J.C., "The violin as a circuit", J. Acoust.Soc.Amer. 35(3), 326-338, 1963. (and unpublished work)

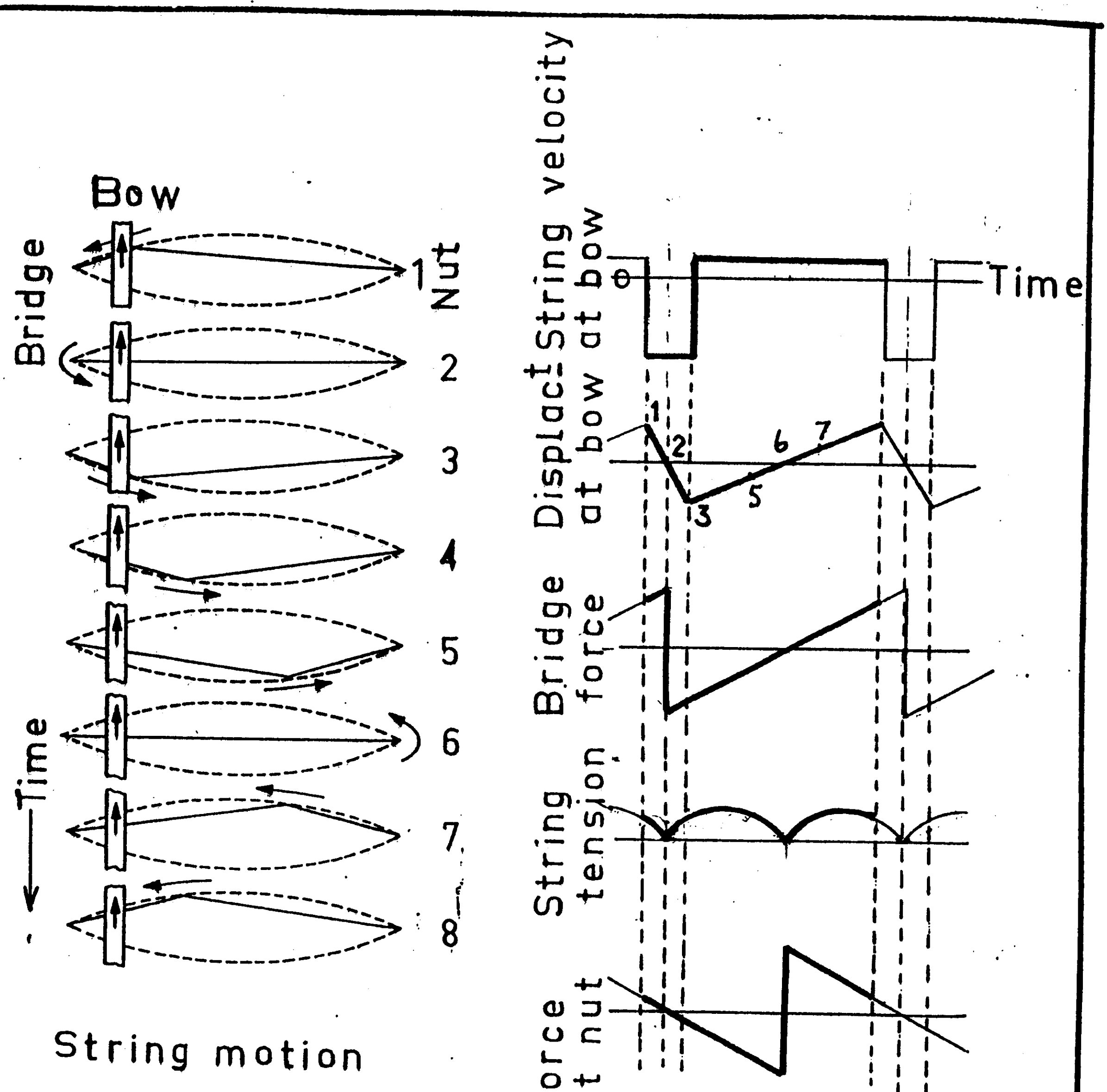
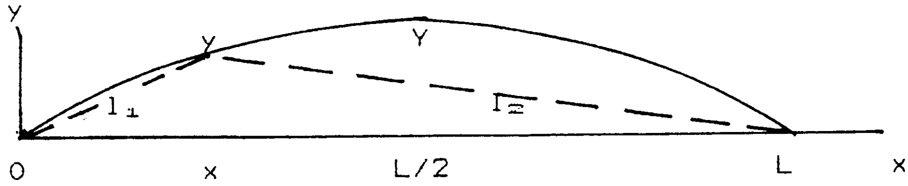
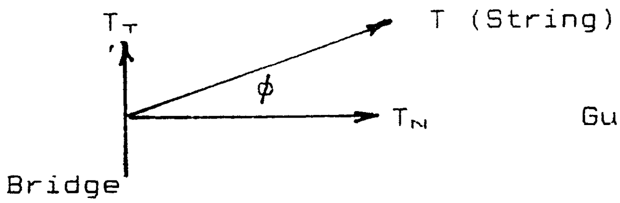


Fig. 1 Bowed string motion and associated effects

JMcL



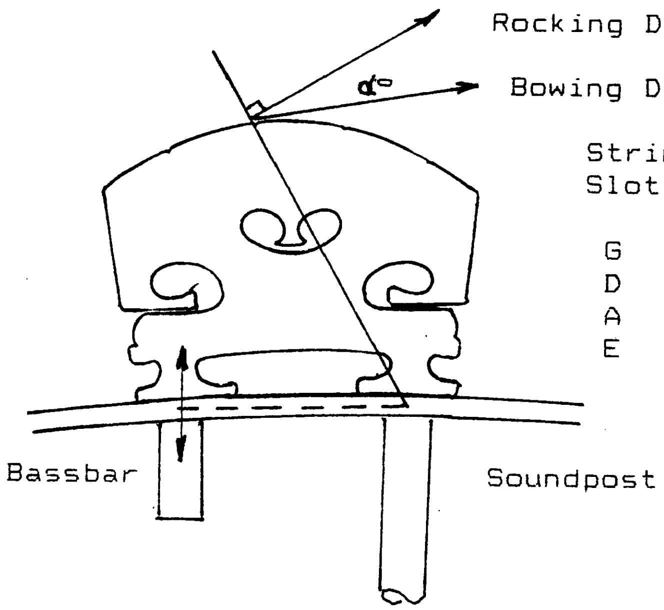
String nomenclature



Gut D string

dia.	0.82 mm
m′	0.64 g/m
D	1.22 g/cc
E_u	4.2E9 N/m^2
S_L	6.7E3 N/m

Figure 2 String geometry



String Slot	α°	cos α	lever ratio
G	20	0.94	46/30
D	25	0.91	41/30
A	30	0.87	36/30
E	30	0.87	29/30

Figure 3 Bridge geometry