Quanta: The geometry of crystalline objects in high dimensions

by Neville Fletcher

Most physicists will be familiar with the expression, put forward by Euler in about 1792 and later proved in detail by Legendre and by Cauchy, giving the relationship between the number of vertices, the number of faces, and the number of edges for an object with crystal-like habit, such as a geometrical polyhedron or a faceted gemstone, namely

$$\text{vertices} + \text{faces} = \text{edges} + 2 \tag{1}$$

or, with a more compact notation,

$$N_V + N_F = N_E + 2. \tag{2}$$

The general validity of this result can be established for any polyhedron by noting that it is correct for the case of a tetrahedron, where $N_V = 4$, $N_F = 4$ and $N_E = 6$. Adding another point vertex near the centre of a face increases N_V by 1, increases N_F by 2 since the original face is removed, and increases N_E by 3. The relation therefore continues to apply. The extra vertex could be lifted above the original face plane, making a completely convex structure, or depressed to make a concave structure, without changing the relationship.

There is a topological restriction applying to (2) in that the space enclosed within the polyhedral surface, as well as the space outside, must be simply connected, by which we mean that it must be possible to shrink any closed curve drawn within this space to zero size without crossing any faces or edges. This rules out objects in which the enclosed space is toroidal, or polygonal tubes with both ends open.

An extended formulation

The primary aim of this note is to show the simple relation (2) can be extended to apply to crystal-like objects in a space of any number of dimensions. It could well be that this question has already been investigated and resolved by mathematicians, but I have no citations to make. Instead I rely upon the approach of the philosopher Ludwig Wittgenstein, who wrote *"I give no sources, because it is indifferent to me whether what I have thought has already been thought before me by others."*

As a first step, equation (2) can be rewritten:

$$N_V + N_F = N_E + N_S + 1, \tag{3}$$

where N_S is the number of unconnected spaces contained within the object. For all the cases we have considered so far, $N_S = 1$, but it is easy to construct an example with a higher value of N_S. Suppose we begin with a simple cube, for which $N_S = 1$, so that it clearly satisfies (2). Now insert a partition parallel to one of the square surfaces of the cube. Because this partition divides each of the edges and faces that it intersects into two parts, this increases N_V by 4, N_F by 5 and N_E by 8, and, because the interior volume is now divided into two sections, increases N_S by 1. The relation (2) is still satisfied, and further divisions of the interior volume have similar results, provided they do not result in any torus-like structures.

If we examine the dimensionality of the objects referred to by terms in equation (3), we see that those on the left-hand side have dimension 0 and 2 respectively, while those on the right-hand side have dimensionality 1 and 3, omitting consideration of the final numerical term. This observation leads us to propose a generalization of (3) of the form

$$\sum_{n=0}^{\infty} N_{2n} = \sum_{n=0}^{\infty} N_{2n+1} + 1 \tag{4}$$

where N_n is the number of elements in the structure with dimensionality n, all elements being assumed to have simple geodesic form — points, straight lines, planes, etc.

It is straightforward to check this relationship in spaces with a small number of dimensions, since $N_n = 0$ if n exceeds the dimensionality of the space. For a zero-dimensional space only point vertices can exist and an object can have only one, so that $N_0 = 1$ and $N_n = 0$ for $n > 0$, thus satisfying (4). Such a zero-dimensional object can, of course, also exist in any space of higher dimensionality. For a one-dimensional space, and neglecting the case of a zero-dimensional object, a simply connected object with m point vertices must have $N_0 = m$, $N_1 = m - 1$ and $N_n = 0$ for $n > 1$, again satisfying (4).

Two-dimensional objects in two-dimensional space become more complex, since they may or may not enclose two-dimensional spaces. A simple cross has $N_0 = 5$, $N_1 = 4$ and $N_2 = 0$, satisfying (4). A square has $N_0 = 4$, $N_1 = 4$, $N_2 = 1$, again satisfying (4), and if we add a line joining two opposite edges then $N_0 = 6$ and $N_1 = 7$, since each of these edges is split into two separate one-dimensional objects, and $N_2 = 2$, again satisfying (4). The familiar three-dimensional case has already been discussed using the notation of equation (3).

Extension to objects and spaces with dimensionality higher than three remains a conjecture, but a steady progression through dimensionality exists. To convert from an object of dimensionality zero to a dimensionality of one in a space of dimensionality one or higher, we must add another point not lying in its zero-dimensional space and join them with on object of dimensionality one (a line). To convert from an object of dimensionality one (a line) to a minimal object of dimensionality two (a triangle), we must connect its vertices to a point not lying within its one-dimensional space. To convert from a triangle to a minimal object of dimensionality three, we must connect its vertices to an additional point not lying in its plane, thus making a tetrahedron. This leads us to surmise that, in order to create a minimal object of dimensionality four, we must take a tetrahedron and connect its vertices to an extra point not lying in the three-dimensional space that it occupies. Unfortunately it is nearly impossible for us to visualize such an object! Once again, following the discussion of three-dimensional objects, the applicability of the result must be restricted to objects with compact topology, eliminating those with the analog of toroidal structure.

Conclusion

This short note takes a well-known theorem relating to the geometry of polyhedral crystalline forms in three dimensions, expresses it in a notation that is applicable in any number of dimensions, and conjectures that this extended version, which is demonstrably correct for objects in spaces of dimensionality three or less, can also be applied to similar objects of higher dimensionality embedded in spaces of higher dimension. With theoretical physics expanding into spaces with dimensions higher than twelve, this conjecture is perhaps worthy of further attention and could even prove useful. (Perhaps string theorists already know all about it!)