Wave Propagation on an Acoustically Perturbed Jet

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Summary

Theories of the initiation and propagation of transverse sinuous disturbances on an air jet emerging from a flue slit into a transverse acoustic velocity field are reviewed and brought into a form suitable for comparison with experiment. Available experimental data is then reviewed and is found to be only moderately well accounted for by the existing theory. A new series of experimental determinations of wave velocity on a jet under conditions relevant to sound generation in musical instruments is then described, these measurements extending over a larger range of frequencies and blowing pressures than earlier measurements. For jets with Reynolds number below about 1000 the wave velocity is found to be constant along the length of the jet and to depend not upon the jet velocity V but rather upon the integral J of V^2 across the width of the jet. At low frequencies, ω, the wave velocity is approximately 0.7 $(J\omega)^{1/3}$, in agreement with existing theory, but at higher frequencies it becomes independent of ω and approximately given by $50\,J$ (where S.I. units are implied). Wave amplification along the jet is also studied and the wave amplitude found to increase like $\exp(\mu x)$ where μ is positive only below a critical frequency ω^*, dependent on the jet parameters. For the jets studied, μ is of order $1 \times 10^3\ \mathrm{m}^{-1}$ below ω^*. A simple theory based on dimensional analysis is proposed to describe these experimental results.

Wellenausbreitung über einen akustisch gestörten Strahl

Zusammenfassung

Die Theorien, die die Entstehung und Ausbreitung transversaler, sinusartiger Störungen in einem aus einem Labialschlitz in ein transversales, akustisches Geschwindigkeitsfeld austretenden Luftstrahl beschreiben, werden gesichtet und in eine für den Vergleich mit experimentellen Ergebnissen passende Form gebracht. Eine Überprüfung der verfügbaren experimentellen Daten zeigt, daß sie nur in bescheidenem Ausmaß durch die bestehende Theorie erklärt werden können. Es werden dann neue Meßreihen zur Bestimmung der Wellengeschwindigkeit in einem Luftstrahl, und zwar unter Bedingungen, die für die Schallerzeugung in Musikinstrumenten relevant sind, beschrieben. Diese Messungen erstreckten sich über größere Frequenz- und Blasdruckbereiche als die bisherigen. Für Strahlen mit einer Reynolds-Zahl unter 1000 findet man, daß die Wellengeschwindigkeit über die Länge des Strahls konstant ist und nicht von der Strahlgeschwindigkeit V, sondern eher vom Integral von V^2 über die Strahlbreite abhängt. Bei tiefen Frequenzen ω beträgt die Wellengeschwindigkeit in Übereinstimmung mit der bestehenden Theorie näherungsweise 0,7 $(J\omega)^{1/3}$, bei höheren Frequenzen aber wird sie unabhängig von ω und ist näherungsweise gegeben durch $50\,J$ (wobei SI-Einheiten impliziert sind). Es wurde ferner die Wellenverstärkung längs des Strahls untersucht. Es zeigte sich, daß die Wellenamplitude mit $\exp(\mu x)$ ansteigt, wobei μ nur unterhalb einer von den Strahlparametern abhängigen kritischen Frequenz ω positiv ist. Für die untersuchten Strahlen war μ unterhalb ω von der Größenordnung $10^3\ \mathrm{m}^{-1}$. Zur Beschreibung dieser experimentellen Ergebnisse wird eine einfache, auf einer Dimensionsanalyse basierende Theorie vorgeschlagen.

Propagation d'ondes sur un jet ayant subi une perturbation acoustique

Sommaire

On passe en revue les théories traitant de la création et de la propagation de perturbations ondulatoires transversales affectant un jet d'air issu d'une fente dans un tuyau et débouchant dans un champ sonore à vitesse acoustique transversale au jet. Après les avoir mises sous une forme prêtant aux comparaisons, on passe en revue les résultats disponibles de travaux expérimentaux et on constate qu'ils ne sont que partiellement expliqués par la théorie. On a donc procédé à de nouvelles expériences pour déterminer la vitesse des ondes sur un jet sous des conditions correspondant à l'émission sonore des instruments de musique et dans des gammes de fréquences et de pressions de soufflage plus étendues que chez les auteurs antécédents. Pour les jets ayant des nombres de Reynolds inférieurs à 1000 environ, la vitesse des ondes en question est constante tout le long du jet et ne dépend pas de la vitesse V du jet, mais plutôt de J, intégrale de V^2 à travers la section du jet. Aux basses fréquences la vitesse des ondes vaut approximativement 0,7 $(J\omega)^{1/3}$ ce qui est en accord avec les théories disponibles, mais aux fréquences plus élevées, elle devient indépendante de la pulsation ω et vaut approximativement $50\,J$ (en unités du système S.I.). On a étudié également l'amplification des ondes le long du jet et on a trouvé que l'amplitude croissait comme $\exp(\mu x)$ où μ est une constante positive pourvu que la fréquence angulaire soit inférieure à une certaine valeur critique ω^* qui est fonction des paramètres du jet. Pour les jets étudiés, μ était de l'ordre de $10^3\ \mathrm{m}^{-1}$ (aux pulsations inférieures à ω^*). Pour rendre compte de tous ces résultats expérimentaux, on propose une théorie simple, fondée sur l'analyse dimensionelle.

1. Introduction

The propagation of waves on a jet disturbed by acoustic or other influences has excited scientific curiosity since the time of Tyndall and Helmholtz, partly because of the intrinsic interest of the phenomena involved and partly because of their importance to the understanding of the operation of certain musical wind instruments such as flutes and organ pipes. In the century or more since that time we have progressed somewhat in our understanding of these matters, but the mathematical analysis of the jet is so formidable a problem that the enlightenment to be gained from more recent treatments is not greatly in advance of that set forth in the classic studies of Rayleigh [1], [2].

Our purpose in the present paper is to review these theoretical studies, for the practically important case of sinuous disturbances on an essentially planar air jet, to review some of the more helpful experimental studies, and finally to present the results of a new set of experiments, together with a theoretical interpretation of those results. In choosing the experimental parameters we have kept within the domain that is of importance in the operation of flutes and organ pipes. We omit discussion of the interesting but less important case of symmetrical ("varicose") disturbances of the jet flow.

2. Basic theory

The work of Rayleigh [1], [2] provides the foundation for understanding the behaviour of a perturbed jet. As a first case he examines the behaviour of an inviscid plane laminar jet of thickness $2l$ moving with velocity V through an unbounded space filled with the same medium. For the propagation of a transverse sinuous disturbance of the jet with complex angular frequency $n = \omega + \mathrm{j}\,\omega'$ and propagation number $k = 2\pi/\lambda$ such that the displacement has the form

$$y = A \exp[\mathrm{j}\,(n t \pm k x)] \tag{1}$$

the dispersion relation is shown to be

$$(n + k V)^2 \tanh k l + n^2 = 0\,. \tag{2}$$

Solving for n and substituting back in eq. (1) shows that the wave on the jet propagates with velocity

$$u = V/(1 + \coth k l) \tag{3}$$

and grows exponentially with time as $\exp(\omega' t)$ or equivalently with the distance x travelled by the wave as $\exp(\mu x)$ where

$$\omega' = k V (\coth k l)^{1/2}/(1 + \coth k l) \tag{4}$$

$$\mu = k (\coth k l)^{1/2}\,. \tag{5}$$

The analysis on which these results are based assumes that the wave amplitude A is less than the half-width l of the jet and it is not known to what extent they can be extrapolated to greater amplitudes.

These equations imply that the jet is unstable at all frequencies. The phase velocity u approaches half the jet velocity at frequencies high enough that $kl \gg 1$, which is equivalent to a requirement that the wavelength become much less than the jet width, while the instability coefficient μ increases without limit with increasing frequency. At frequencies low enough that $kl \ll 1$, eqs. (3) and (5) give

$$u \approx k l V = (l V \omega)^{1/2}\,, \tag{6}$$

$$\mu \approx (k/l)^{1/2} = l^{-3/4} V^{-1/4} \omega^{1/4}\,. \tag{7}$$

This analysis is sufficient to give a qualitative description of the behaviour of a jet emerging from a narrow flue into an acoustic field as was shown by Fletcher [3]. Suppose that the acoustic flow velocity normal to the jet has the value $v \cos \omega t$ everywhere in the plane of the jet. This has the effect of displacing the jet bodily in the y direction by an amount $(v/\omega) \sin \omega t$ everywhere except at the origin, where the existence of the flue slit maintains the displacement at zero. This has the same effect as superposing on the bodily displacement of the jet a localized displacement

$$-(v/\omega) \sin \omega t$$

at the origin, which then propagates in both directions and grows according to eqs. (3) and (5). Combining all these effects leads to a jet displacement at position x and time t of

$$y(x, t) = (v/\omega)\,\{\sin \omega t - \cosh \mu x \times \times \sin[\omega (t - x/u)]\}\,. \tag{8}$$

This expression predicts a jet shape at an instant of time that is in quite good qualitative agreement with photographs of organ-pipe jets. It is also in agreement with the expectation that a slow transverse flow should simply deflect the jet in the direction of flow since, for low enough frequencies and small enough distances that $\mu x \ll 1$ and $\omega x/u \ll 1$, eq. (8) can be expanded to give

$$y(x, t) \to (v x/u) \cos \omega t\,. \tag{9}$$

The weakness of this treatment when applied to real jets was realised by Rayleigh. The problem is manifest by the fact that the growth constant μ diverges as ω approaches infinity so that the jet is predicted to be wildly unstable. Clearly some other effect must intervene to limit this instability and the next generation of theories takes up this question.

3. Refined theory

The physical feature omitted from the basic theory is the viscosity of the fluid. Only for an inviscid fluid can a jet maintain two surfaces of separation across which the velocity changes discontinuously. In a real fluid there will be a velocity gradient across a transition layer of finite extent. Expressed in other words, instead of two vortex sheets, the vorticity

$$Z = \tfrac{1}{2}\,\mathrm{d}V/\mathrm{d}y$$

will be finite in the whole jet environment.

An exact solution of the flow behaviour of even the undisturbed jet involves solution of the Navier-Stokes equations for the problem. This is extremely complicated but a simplified solution based on the Prandtl boundary layer equations has been given by Bickley [5].

A real jet is characterized not so much by its central velocity $V(0)$ as by its total momentum flux per unit length of flue slit

$$M = \varrho J = \varrho \int_{-\infty}^{\infty} V^2\,\mathrm{d}y \qquad (10)$$

which is constant across all planes along the jet. The integral J is more convenient for our later development than is M. In terms of the boundary layer equations Bickley was then able to show that, for a jet issuing from a slit of infinitesimal width at $x = 0$,

$$V(y) = 0.4543\ldots(J^2/\nu x)^{1/3}\,\mathrm{sech}^2(y/b) \qquad (11)$$

where

$$b = 3.635\ldots(\nu^2/J)^{1/3}x^{2/3} \qquad (12)$$

and ν is the kinematic viscosity ($\nu \approx 1.5\times10^{-5}\,\mathrm{m^2s^{-1}}$ for air). The width b of such a jet increases as $x^{2/3}$ while its central velocity decreases as $x^{-1/3}$. The velocity profile is shown in Fig. 1., where we have also shown the profile of $V(y)^2$, which is sometimes easier to measure. Real jets often conform quite well to the expectations of this theory, as we shall see later.

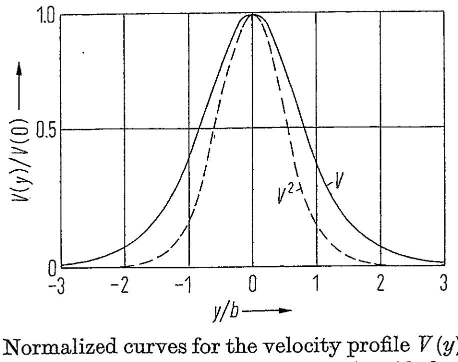


Fig. 1. Normalized curves for the velocity profile $V(y)/V(0)$, and the square of this quantity, in terms of the width parameter b, for a Bickley jet as described in eq. (11).

Rayleigh [2] did not go so far as to consider the detailed behaviour of a jet with a practical profile such as this, but did investigate the behaviour of a jet whose velocity profile was divided into a number of linear segments corresponding to layers of constant velocity gradient (constant vorticity). He was able to show that the existence of unstable solutions ($\mu > 0$) was associated with the presence of an inflection ($\mathrm{d}^2V/\mathrm{d}y^2 = 0$) in the velocity profile. He also found that, for reasonably jet-like profiles, the instability parameter μ is positive in the low-frequency limit, increases with increasing frequency to a maximum when $kb \approx 1$, b being some measure of the jet half-width, and then decreases to become negative for $kb \gtrsim 2$. This is very much the sort of behaviour one would expect intuitively for real jets. Rayleigh's analysis omitted viscosity effects, except in so far as they were supposed to determine the jet profile, but perhaps this is their most important role.

Theoretical developments since that time have been summarized by Drazin and Howard [6]. The general approach has been to assume a velocity profile based on analysis such as that of Bickley, or on a mathematically more simple approximation to it, and then to investigate the wave dispersion relation (and hence the stability of the flow) neglecting both viscosity and the variation of the profile along the jet.

One particular well-known calculation is that of Savic [7] which applies to the Bickley jet profile (eq. (11)), again neglecting viscosity. His result for wave velocity u, expressed in terms of J as defined in eq. (10), is

$$u = 1.016\ldots(J\omega)^{1/3} \qquad (13)$$

for neutrally stable disturbances, a result which he found to describe fairly adequately the motion of the vortex streets photographed on jets by Brown [8]. The analysis gave no direct information about the growth parameter μ.

More recent studies of the same jet profile, again in the inviscid approximation, have been made by other workers [9] and Drazin and Howard [6] quote the results of a calculation by Lessen and Fox [10] which is shown in Fig. 2 in a form recalculated to use the formalism of our discussion. The growth parameter μ for sinuous disturbances rises with increasing frequency to pass through a broad maximum near $kb = 0.6$, where b is the width scale of the jet, then falls to zero when $kb = 2$. This behaviour thus mirrors that found by Rayleigh for his simple slab models and has no

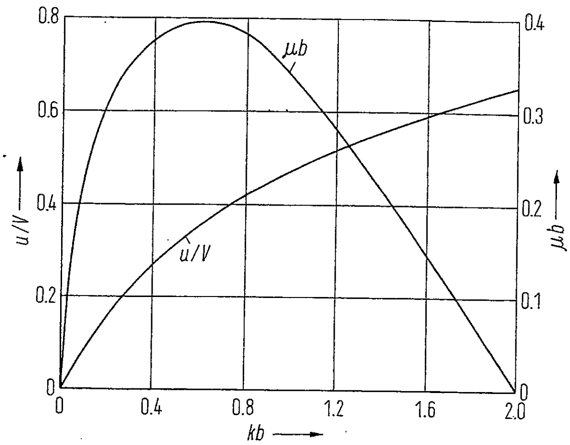


Fig. 2. The wave velocity $u = V c_r$ and the amplification factor $\mu = k c_i / c_r$ as functions of the frequency parameter kb for a jet of Bickley profile, as deduced from the calculations reported by Drazin and Howard [6]. V is the centre velocity of the jet, b the width parameter, and $k = \omega/u$ where ω is the angular frequency. In terms of the original analysis, c_r is the real part and c_i the imaginary part of the normalized wave velocity u/V and $kb = \alpha$.

divergence of the instability at high frequencies. For practical purposes the curve can be represented by the equation

$$\mu b \approx 0.74[1 - \exp(-3kb)] - 0.37\,kb \tag{14}$$

for $0.1 \leq kb \leq 2$. There is some uncertainty in the behaviour for $kb < 0.1$ and μb may not go to zero as $kb \to 0$.

When we consider the phase velocity of the disturbance we find that, within the range $0.4 \lesssim kb \leq 2$ the velocity u is well approximated by the relation

$$u \approx 0.65\,(J\omega)^{1/3} \tag{15}$$

which is essentially Savic's result (eq. (13)), while in the range $0 \leq kb \lesssim 0.4$

$$u \approx 0.82\,(J\omega/V)^{1/2} \approx 0.95\,(b\,V\omega)^{1/2} \tag{16}$$

which is essentially Rayleigh's result (eq. (6)). For practical purposes it is sufficient simply to take whichever of the results (15) or (16) predicts the lower velocity. This approach to the Rayleigh result at low frequencies is very much what we should expect on general grounds. The behaviour of u as a function of kb is also shown in Fig. 2.

It is a matter of conjecture to what extent the proper inclusion of viscosity in the jet equations might modify these conclusions, since no detailed treatment seems to have been published. From the form of the Navier-Stokes equations one might surmise that the effects of viscous forces should be greater at high than at low frequencies, but it is by no means impossible that their inclusion might alter the functional form of equations such as (13).

4. The spreading jet

One of the clear effects of viscosity which can be relatively simply appreciated relates to the spreading of the jet as described by Bickley [5] and set out in eqs. (11) and (12) above. It is therefore appropriate to examine wave propagation behaviour on a spreading jet for a particular disturbing frequency within the range to which the jet is sensitive.

Suppose the jet emerges from a narrow flue slit of width $2l$ under the action of a blowing pressure p_0. If the channel of the slit is short then the jet velocity profile will be of top-hat shape with a velocity V_0 given by

$$p_0 = \tfrac{1}{2}\varrho\,V_0^2 = \varrho J/4l \tag{17}$$

where ϱ is the density of air and J is defined by eq. (10). If the channel is longer then the velocity profile will tend to assume the parabolic Poiseuille form with a central velocity and J value rather smaller than that given by eq. (17). After emerging into the free air however, the jet will rapidly assume a Bickley profile as given by eqs. (11) and (12) except that the effective origin of the x-coordinate will be displaced backwards somewhat behind the slit exit.

For any given disturbing frequency ω the jet displacement will now obey an equation like (8) with the modification that both the amplification parameter μ and the wave velocity u will depend upon the coordinate x along the jet in a manner determined by the width and central velocity of the jet, as discussed in the previous section.

Taking the wave velocity u first, our discussion showed that this is determined essentially by the one of eqs. (15) and (16) that predicts the lower velocity, the transition occurring near $kb \approx 0.4$. From these equations we easily find that the transition from Rayleigh to Savic behaviour should occur near an angular frequency

$$\omega \approx 0.2\,V/b \tag{18}$$

where V is the central velocity. Using Bickley's equations (11) and (12) we find that the transition should occur near the position

$$x_c \approx 0.02\,J/\nu\,\omega \tag{19}$$

where the coordinate x is measured from the apparent origin of the jet at a distance

$$x_0 \sim 0.14\,J^{1/2}\,l^{3/2}/\nu \tag{20}$$

behind the slit, $2l$ being the slit width and ν the kinematic viscosity of the fluid. J is given by eqs. (10) and (17).

For $x > x_c$ we expect the velocity to behave as given by eq. (15) which means that, since J is constant along the jet, u will also be constant downstream from the transition point. For $x < x_c$ eq. (16) applies so that the wave velocity decreases progressively as we approach the slit. Using eqs. (11), (12), (15) and (16) we can write

$$u \approx 1.2\,(J\,\nu\,\omega^3)^{1/6}\,x^{1/6} \quad \text{for} \quad x < x_c \tag{21}$$

$$u \approx 0.65\,(J\,\omega)^{1/3} \quad \text{for} \quad x > x_c \tag{22}$$

with the transition point x_c again being determined by the intersection of these two expressions. Again in eq. (21) we must measure x from the notional origin distant x_0 behind the slit, as given by eq. (20). J is simply evaluated in terms of slit-width and blowing pressure from eq. (17).

The final point to be considered is the expected behaviour of the amplification factor μ, given by eq. (14), for a spreading jet. This expression is given in terms of kb which, with the aid of eqs. (12), (21) and (22), can be written

$$kb \approx 3.0\,(\nu\,\omega\,x/J)^{1/2} \quad \text{for} \quad x < x_c \tag{23}$$

$$kb \approx 5.6\,(\nu\,\omega\,x/J)^{2/3} \quad \text{for} \quad x > x_c\,. \tag{24}$$

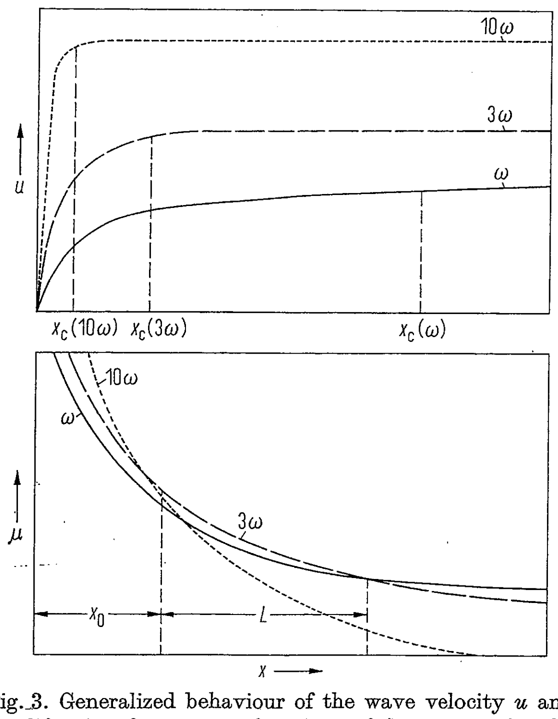


Fig. 3. Generalized behaviour of the wave velocity u and amplification factor μ as functions of distance x, for the case of a Bickley jet emerging from a slit of infinitesimal width at $x = 0$, for three different frequencies. The curves are based on the calculations reported by Drazin and Howard [6] and the coordinate x_c, given by eq. (19), shows the point of transition between the approximations (21) and (22) or (23) and (24). For the case of a jet of length L emerging from a slit of finite width at effective coordinate x_0, the gain parameter G of eq. (25) is equal to the area under the appropriate μ curve between x_0 and $x_0 + L$.

This general behaviour is illustrated in Fig. 3 for several different frequencies, both for the wave velocity u and the amplification factor μ.

A measure of the total gain along a jet of length L is given by

$$G = \int_{x_0}^{x_0+L} \mu(x)\,\mathrm{d}x \tag{25}$$

and therefore is the area under these curves between x_0 and $x_0 + L$, where x_0 is the distance of the slit from the effective jet origin, as given by eq. (20). Clearly for every jet length L (for fixed J and x_0) there is a particular frequency for which G is a maximum, decreasing for both higher and lower frequencies. This optimum frequency is high for short narrow jets and, conversely, low for long broad jets.

It would seem from the analysis that μ becomes negative for $kb > 2$ so that the wave might be expected to decay as it proceeds past this point on the jet, as given by eq. (24). This may well occur but we must be wary of too simplistic an interpretation of the equations since a more complex analysis may well be necessary once the wave amplitude becomes large.

5. Previous experiments

Although the number of experimental studies of acoustically perturbed jets is very large, few of these have been directed towards understanding the specific propagation phenomena discussed here. They have been concerned, rather, either with exploring the boundaries of the regions of stability for propagation on the jet, or else with aspects of particular jet phenomena such as edge tones or the operation of organ pipes.

The main exceptions are found in the work of Brown [8], Sato [11], Chanaud and Powell [12] and Coltman [13] which we now review briefly.

Brown's studies [8] were concerned largely with the generation and motion of vortices on jets perturbed by acoustic signals. These vortices, which were studied photographically, are the end-product of the perturbation when the wave amplitude becomes larger than its wavelength. The slit widths used ranged from 0.25 to 4 mm, the jet velocity from 1 to 20 m s^{-1} (corresponding to blowing pressures from 0.5 to 200 Pa) and frequencies from 100 to 400 Hz. These represent a very useful series of measurements which were found by Savic [7] to conform generally to the predictions of his theoretical expression (13). There is, however, a question whether or not the propagation velocity of the fully developed vortices is the same as the wave velocity for sinuosities on the jet.

Sato [11] in his experiments examined flow from much wider slits, 4.4 to 40 mm, and generally at rather higher velocities, 2 to 20 m s^{-1}. He paid much greater attention to the velocity profile of the jet, showing that it did not closely approach the Bickley form until a considerable distance downsteam from the slit. His study was more concerned with stability and the growth of fluctuations on the jet than with the simple propagation of waves, but his measurements, using a hot-wire anemometer, show behaviour of the general form illustrated in Fig. 3. Those studies were later extended by Sato and Sakao [14] so slits of width 0.2 and 1.1mm, again concentrating largely upon identification of domains of stability and of instability.

The work of Chanaud and Powell [12] is similarly concerned mainly with finding the conditions under which a jet is unstable rather than with studying wave propagation. The experiments made use of a water jet rather than an air jet.

The experimental studies of Coltman [13] are those most closely related both to the theory we have discussed above and to our own experiments to be described in the next section. He used a Pitot-tube sensor and a phase measurement system to determine the propagation velocity of transverse waves on an air jet and was also able to determine their relation in phase to the acoustic disturbance. Transverse sweeps with the Pitot-tube and its associated microphone also allowed the velocity profile and its displacement in the acoustic field to be examined.

For a slit of width 1.59 mm and with the flue passage in the form of a long parallel-walled channel, Coltman found that his jet emerged from the slit and travelled with almost negligible spread for a distance of at least 7 mm. The influence of the acoustic field was simply to displace the jet sideways without change of profile as waves propagated along the jet. The lack of spread in this case might perhaps be expected to make the jet behaviour rather more like a Rayleigh jet than a Bickley jet in this near-slit region.

Coltman's results, redrawn in Fig. 4a, show that, when the measurement distance is more than about half a wavelength from the slit, the variation of phase shift with frequency is linear, implying a constant phase velocity u independent of frequency. Calculation from the curves shows that u is between 0.44 and 0.5 times the jet velocity. Both these conclusions are in good agreement with Rayleigh's eq. (3) above.

For distances less than about half a wavelength from the slit the curves of Fig. 4a suggest higher phase velocities. From eq. (8) however, this does not imply an increase in the value of u but rather exhibits the effect of the phase shift of $\pi/2$ in the displacement y as we pass to the limit given by eq. (9).

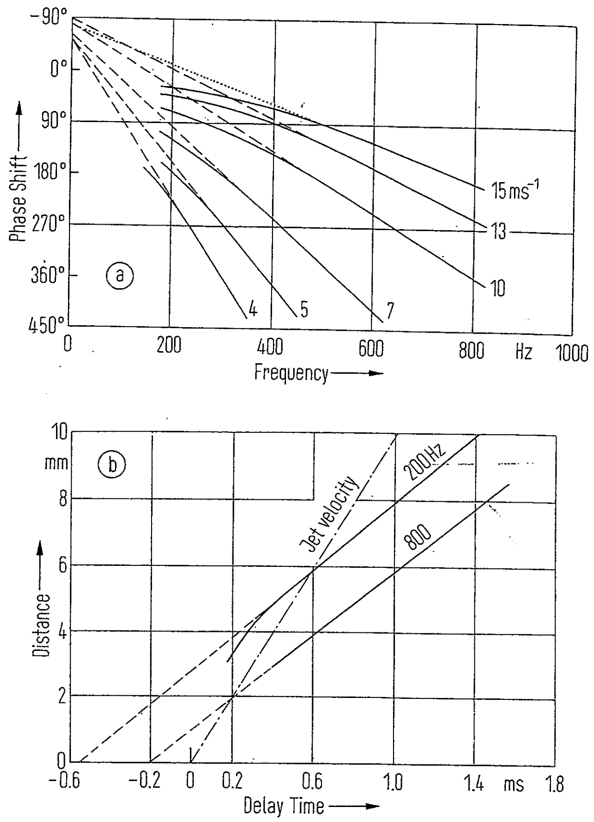


Fig. 4. (a) Phase shift as a function of frequency for a given jet length and for various jet speeds and (b) propagation delay time as a function of distance for two different frequencies, for waves on a jet as determined by Coltman [13]. The linear parts of the curves have been extrapolated backwards and, to a first approximation, intersect the axis near (a) a phase of $-90°$ or (b) a delay time of $-T/4$ where T is the periodic time. Curves for low jet speeds have rather smaller intercepts, as discussed in the text, and the extrapolation shown dotted in (a) is suspect.

If we extrapolate the straight line portions of the curves of Fig. 4a back to zero frequency, as is shown in the diagram, then they intersect the axis at a phase shift which lies between $-50°$ and $-90°$. This corresponds to a displacement in the travelling wave on the jet at the slit which lags by this amount behind the acoustic velocity and is therefore nearly in phase with the acoustic displacement as is required by the analysis leading to eq. (8). The expected extrapolation intercept on this basis is $-90°$ but this can only be achieved if the phase velocity u remains constant down to zero frequency. In fact we expect from eqs. (3) or (6) that u will decrease at low frequencies and this will have the effect of moving the intercept closer to 0°. The slightly irregular behaviour of the intercept of the extrapolation is perhaps due to the limited range of curve shown which leads to an incorrect assessment of the slope of the linear part of the curve.

A complementary measurement in which distance along the jet is varied rather than frequency, as shown in Fig. 4b, leads to similar conclusions. Propagation velocity is constant along the jet except close to the origin where the phase shift implied by eq. (8) leads to an apparent increase in phase velocity. The curve for 800 Hz extrapolates to $-T/4$ as expected, where T is the oscillation period, but that for 200 Hz extrapolates to about $-T/8$, perhaps as a consequence of a decrease in the real propagation velocity u, as opposed to the measured phase velocity, very close to the flue slit. It is not possible to be more definite in these interpretations without a much more detailed study.

It is interesting that Coltman's jet behaves in such good conformity with the predictions of Rayleigh's theory despite the influence of viscosity and the inevitable diffusion of the shear layer. This may be due in part to the particular velocity profile produced by the slit he used but it is also true that the predictions of eqs. (15) and (16) for the Bickley profile are not very different from those of Rayleigh over much of the range of the measurements. It may be that the effects of viscosity are such as to bring the two treatments into even closer agreement. It is this question which our own experiments were designed in part to answer.

6. Measurement of phase velocity

For a new measurement of the phase velocity of sinuous disturbances on an air jet we used essentially the method described by Coltman [13]. Our objective was not so much to produce results of high accuracy as to explore, with such accuracy as was easily attainable, a reasonably large area of musically significant jet behaviour. With this end in view, no particular precautions were taken to exclude extraneous sound or to enhance the stability of the jet flow above that normally found in wind instruments.

The jet was defined by the parallel passage between two perspex blocks fixed between two perspex sheets as shown in Fig. 5. The separation of the blocks could be varied from zero to several millimeters, thus defining the jet thickness $2l$ at the flue exit. The jet width, defined by the distance between the two perspex sheets, was about 50 mm and the length of the passage between

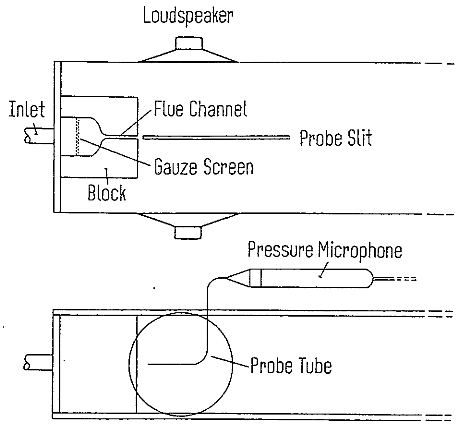
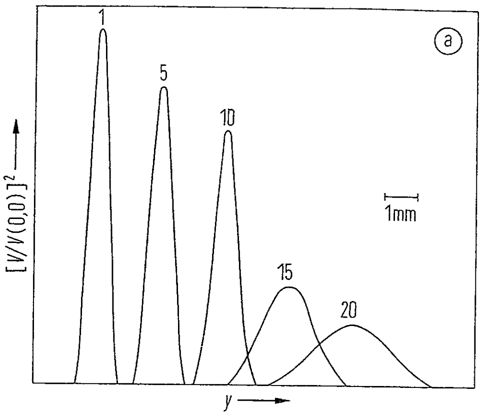


Fig. 5. The experimental arrangement in plan (with the probe microphone omitted) and in elevation.

the two blocks was about 20 mm. The jet was fed with air or nitrogen from a gas cylinder, and a wire mesh was interposed between the inlet pipe and then entry to the flue passage to stabilize the flow.

Blowing pressure could be measured at the inlet to the flue slit using an electronic manometer. Alternatively the momentum flux per unit area, $M/2l$, defined by eq. (10), and hence the integral J, could be measured directly in terms of the stagnation pressure p_0 at a rectangular Pitot tube exactly fitting the slit at its exit end. In the absence of viscous losses these two quantities are related by eq. (17), as was verified for wide slits. For narrow slits viscous losses in the channel reduce the value of J at the exit so that our results are reported either in terms of the measured J at the exit or in terms of an effective blowing pressure p_0 related to J by eq. (17).

A narrow Pitot tube with a rectangular entrance of width about 0.2 mm was also used to study the variation of the jet velocity V (measured directly as V^2) both in the x direction along the jet and in the y direction transverse to it. For this purpose the traversing screw carrying the Pitot tube drove a multi-turn potentiometer so that a direct plot of V^2 against x or y was produced on an XY recorder. Some typical measurements are shown in Fig. 6, from which it is clear that the profile and velocity behaviour only roughly approximates that of a Bickley jet. The principal deviation from this behaviour is that the jet slows and broadens at first more slowly and then more rapidly with distance than is predicted by eqs. (11) and (12). After travelling perhaps 15 to 20 mm from the flue, the jet becomes broad and diffuse, presumably from the development of turbulent instabilities. For this reason all our measurements were confined to the first 15 mm of jet travel.



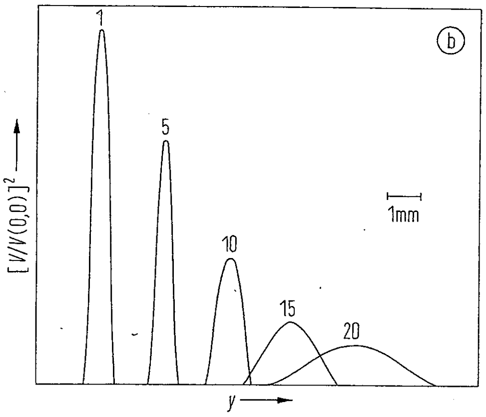


Fig. 6. Profiles of the normalized squared velocity $[V(x, y)/V(0, 0)]^2$ as measured with a fine Pitot tube for distances x along the jet, shown in millimetres as a parameter. In each case the effective blowing pressure p_0 is 70 Pa and the slit widths are (a) 1 mm and (b) 0.5 mm.

An acoustic field to deflect the jet was provided by two small loudspeakers mounted one on either side of the jet plane and connected in antiphase. The balance between the two loudspeakers, which were driven from an audio oscillator through a stereo amplifier, could be adjusted to give an approximate pressure null (and thus a velocity maximum) in the plane of the jet.

The behaviour of the jet was examined by loading it with smoke (incense provided a more pleasant and less messy alternative to the more usual tobacco smoke) and illuminating it stroboscopically. The jet behaviour over the first 15 mm or so of travel was exactly as expected from earlier studies, with sinuous disturbances propagating along the jet and growing with distance to give a deflection of a few millimetres at 10 mm distance, for the acoustic levels used in the measurements.

Measurements of phase velocity for sinuous disturbances on the jet were carried out using a fine Pitot tube connected to a condenser microphone (Bruel and Kjaer probe microphone). With the probe tube located on the jet centre plane and facing towards the flue slit a pressure maximum was observed for each zero crossing of the jet displacement and thus at the second harmonic of the acoustic disturbing frequency. An appropriate filter was used to remove noise from the signal. The Pitot tube was then traversed along the median plane of the jet in its flow direction and the change in phase of the second harmonic signal plotted as a function of distance. The signal was generally clean enough to give good phase information over a range from about 3 mm to 15 or 20 mm from the flue exit.

The first point of interest is that, to within the accuracy, the phase changed uniformly with distance along the whole length of jet over which measurements could be made. This finding, which confirms that made by Coltman [13] away from the immediate vicinity of the flue slit, means that relatively accurate values of phase velocity can quite easily be determined. It also implies that the parameter determining phase velocity in this well developed part of the jet is apparently the integral J defined in eq. (10), for it is only this combination of jet width b and centre velocity V which is constant along the propagation path.

It was found possible to make measurements conveniently for slit widths from 0.5 to 1 mm, for acoustic frequencies from about 150 to 1500 Hz, and for effective blowing pressures from close to zero to an upper limit a little over 100 Pa (1.0 cm water gauge). For blowing pressures higher than this or slit widths greater than about 1 mm the jet behaviour changed significantly and the signal became obscured by turbulence. This is perhaps not unexpected since the effective Reynolds number for the flow is then about 1000 and the perturbing influences are considerable. This transition did not appear to depend greatly upon the width of the slit, for slits narrower than 1 mm, nor was it possible to extend the range of steady behaviour by greatly increasing the length of the flow channel.

The results of measurements of phase velocity carried out in this way are shown in Fig. 7. We see that the phase velocity approaches zero in the limit of low frequencies, rises with increasing frequency and then approaches a limiting value which is maintained to the upper limit of measurement. These results are rather different from the predictions (15) and (16) of the inviscid theory for the Bickley jet but show some general resemblance to Rayleigh's result (4) for a simple jet. Our measurements do not quite overlap the range covered by Coltman [13] but indicate, for the conditions of his experiment, a propagation velocity which is

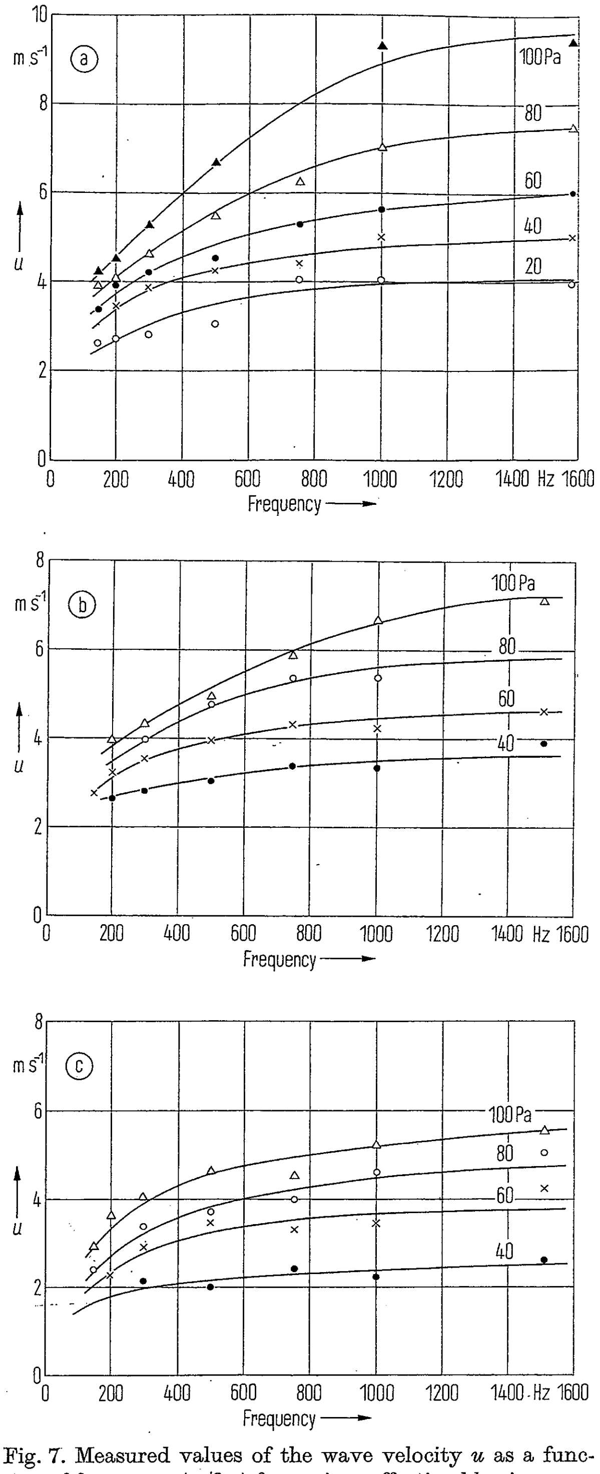


Fig. 7. Measured values of the wave velocity u as a function of frequency $(\omega/2\pi)$ for various effective blowing pressures p_0, shown in pascals as a parameter, for slit widths $2l$ of (a) 1 mm, (b) 0.7 mm and (c) 0.5 mm.

nearly independent of frequency above about 200 Hz and which has a numerical value close to half of the central velocity of the jet at the flue exit. The two studies are thus reasonably consistent.

7. Analysis of measurements

In analyzing the results of our experiments we must bear in mind the theoretical prediction that the propagation behaviour of waves on the jet is considerably influenced by the exact form of the jet velocity profile. We expect this velocity profile to be affected by the length, shape and smoothness of the channel forming the jet and also by the absolute value of the jet velocity. It is, however, not with these details of behaviour that we are principally concerned here but rather with the broad pattern of propagation behaviour over a considerable range of conditions. In this spirit, therefore, our analysis concentrates on general trends of behaviour, leaving aside for future study the irregularities that occur in some particular cases.

The simplest feature of the experimental results to analyze is the plateau value of the phase velocity at high frequencies. We have already remarked that the constancy of propagation velocity along the jet suggests an analysis in terms of the momentum flux integral J of eq. (10), and Fig. 8 therefore

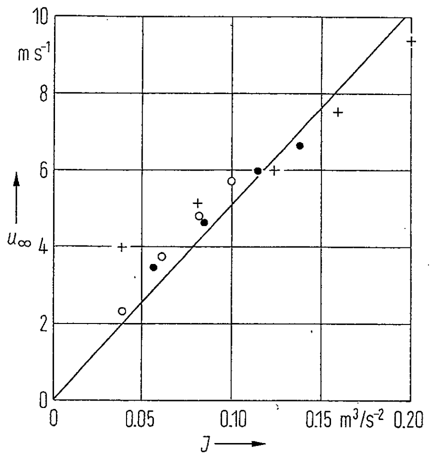


Fig. 8. Limiting wave velocity u_∞ as a function of jet momentum flux parameter J for various slit widths: ○○○ 0.5 mm, ●●● 0.7 mm, +++ 1.0 mm.

shows this limiting velocity, which we denote by u_∞, plotted against J. Clearly the relationship is linear to a very good approximation and is independent of the slit width $2l$, which is as we should expect. The line of best fit to the data in Fig. 8 does not pass exactly through the origin but this seems physically unlikely and could be caused by a small systematic error in the pressure sensing system. The line of best fit through the origin gives, with an accuracy of about ± 10 percent,

$$u_\infty \approx 50\,J \tag{26}$$

where S.I. units are used for u_∞ and J.

The second feature to be analyzed is the behaviour at low frequencies, corresponding to the curved portions of the characteristics of Fig. 7. Clearly these velocity curves approach u_∞ at lower frequencies for small values of J than for large J, so that the behaviour should be expected to be simple only for reasonably large J or for very low frequencies.

In fact the highest pressure curves for the 1.0 and 0.7 mm slits of Fig. 7a and b, which have the highest values of J, show a variation of u quite closely as $\omega^{1/3}$, while a plot of $\ln u$ against $\ln p_0$ in the same figures at a frequency of 200 Hz exhibits a variation as $p_0^{1/3}$, or equivalently as $J^{1/3}$. In this region we find, with an accuracy of about ± 10 percent,

$$u \approx 0.7\,(J\,\omega)^{1/3} \tag{27}$$

at least for $1000 < \omega < 6000$ rad s^{-1}, $0.1 < J < 0.3$ m^3 s^{-2}, which is in good agreement with the theoretical result (eq. (15)).

For higher values of ω or lower values of J the exponent of J increases towards unity and that of ω decreases towards zero until the form (26) is approached. The curves for the 0.5 mm slit and the lower curves for the larger slits show this trend. Because inaccuracies of phase shift measurement over the limited length of jet available made it difficult to extend measurements below 150 Hz, it was not possible to verify that the form (16) was approached in the limit as $\omega \to 0$.

As we remarked before, a proper analysis of wave propagation on a jet involves solution of the Navier-Stokes equation for the situation in which the wave amplitude is larger than the jet width. This is far too complex a problem to attempt here. We can surmise, however, that the reason why the results of measurement differ from the predictions of the simple inviscid theory at high frequencies is in some way connected with the neglect of viscous effects. We initially neglect wave amplitude as a significant parameter. With this in mind we can make some progress by simple dimensional analysis.

To this end, let us consider wave propagation at frequency ω with velocity u along a portion of the jet characterized by a central velocity V and a width parameter b. If the kinematic viscosity of the medium is ν, then the equation relating these quantities can be written in non-dimensional form as

$$\frac{u}{V} = F\left(\frac{V b}{\nu}, \frac{\omega b}{V}\right) \equiv F(\alpha, \beta) \tag{28}$$

where F is an unknown function of the two parameters $\alpha = V b/\nu$, which is essentially the Reynolds number, and $\beta = \omega b/V$ which is related to the Strouhal number which appears as kl in Rayleigh's theory.

We now make use of the observational fact that u does not depend upon b or V individually but only in the combination $b\,V^2$ which is equivalent to J. This can be achieved if

$$F(\alpha, \beta) = \alpha\,F_1(\beta^{1/3}/\alpha) \tag{29}$$

where F_1 is another unknown function, for eq. (28) then becomes

$$u = (J/\nu)\,F_1(\nu\,\omega^{1/3}/J^{2/3}) \equiv (J/\nu)\,F_1(\gamma)\,. \tag{30}$$

The function $F_1(\gamma)$ could, of course, involve fractional powers of its argument γ but, if we assume it to be essentially linear for small values of γ, then eq. (30) becomes

$$u = c_1\,(J\,\omega)^{1/3} \tag{31}$$

where c_1 is a constant which is independent of the kinematic viscosity ν, provided both ν and ω are small, or in particular in the limit as $\nu \to 0$. This is exactly the form of the Savic result (13) or (15) and of our experimental result (27).

We can only guess at the form of $F_1(\gamma)$ for larger values of γ, but the saturation behaviour of the curves of Fig. 7 suggests something like

$$F_1(\gamma) = c_1\,\gamma/(1 + c_2\,\gamma) \tag{32}$$

or

$$F_1(\gamma) = (c_1/c_2)\tanh c_2\,\gamma \tag{33}$$

where c_1 and c_2 are constants. Both of these expressions behave like $c_1\gamma$ if $\gamma \ll c_2^{-1}$, while for high frequencies, provided the viscosity is non-zero, $\gamma \gg c_2^{-1}$ and we have $F_1(\gamma) \to c_1/c_2$ and

$$u_\infty = (c_1/c_2)\,(J/\nu) = \text{const} \times J \tag{34}$$

in agreement with the experimental result (26). Comparison of eqs. (31) and (34) with eqs. (26) and (27), inserting $\nu \approx 1.5 \times 10^{-5}$ m^2 s^{-1}, gives

$$c_1 \approx 0.7\,, \quad c_2 \approx 1000\,. \tag{35}$$

The transition between eqs. (31) and (34) occurs near a frequency ω_c for which $\gamma \approx c_2^{-1}$ or

$$\omega_c \approx J^2/(c_2\,\nu)^3 \approx 3 \times 10^5\,J^2 \tag{36}$$

where the second form applies to air and is in S.I. units. This is in at least semi-quantitative agreement with the curves of Fig. 7. If the kinematic viscosity ν goes to zero, as in the theories of section 3 above, then eq. (31) applies at all frequencies, in agreement with Savic's result.

It is not easy to determine the exact behaviour of μ from experiment, since it probably depends on the jet width b as well as on frequency ω. We can, however, get an approximate measure of its magnitude and frequency variation by measuring the increase in displacement amplitude along a length of jet close to the flue where b does not vary greatly.

Using the general form (8) for the jet deflection, together with the definition (25), we expect for the deflection amplitude at a distance L from the flue the value

$$y(L) \approx (v/\omega) \exp(G) \tag{37}$$

where v is the acoustic particle velocity amplitude at the flue. For a jet length L, if b and hence μ can be taken as nearly constant, $G \approx \mu L$ so that a measurement of $y(L)$, v and ω serves to determine μ.

For a fixed L the pressure amplitude measured by our probe tube reaches a plateau value when the jet deflection amplitude is about twice the jet half-width b. For a given frequency we therefore simply increase the acoustic field until the probe signal reaches this plateau, giving $y(L) \approx b$. A small error in this determination is not very significant. To determine the acoustic velocity amplitude v at the flue we disconnect one loudspeaker and measure the acoustic pressure amplitude p at the flue, using a calibrated condenser microphone. We then have

$$v = 2p/\varrho c \tag{38}$$

where ϱ is the density and c the velocity of sound in air and the factor 2 allows for the effect of the second loudspeaker.

The results of these measurements are shown in Fig. 9 for three different jet situations. In each case a jet length L of 8 mm was used and the effective value of b for the 0.5 mm slit was about 0.45 mm and that for the 1 mm slit 0.7 mm, as measured 5 mm from the flue. Propagation and growth behaviour was normal in each case up to a critical frequency, shown in the figure by an appropriate arrow, above which the probe signal dropped nearly to zero, indicating the absence of a growing wave.

When the measurements of Fig. 7 are used to evaluate the quantity kb for each jet, we find that propagation and growth ceases for values of kb lying between about 1.2 for the 1 mm jet and 2.5 for the slower of the two 0.5 mm jets. In view of the averaged nature of the measurements this represents quite good agreement with the theoretical limit $kb = 2.0$.

The shape of each curve is quite similar but differs from the theoretical predictions shown in Fig. 2. The measured μ initially rises with increasing frequency so that μb has a value close to the predicted peak of 0.4 near $kb = 0.5$. Above this the experimental values decrease slightly,

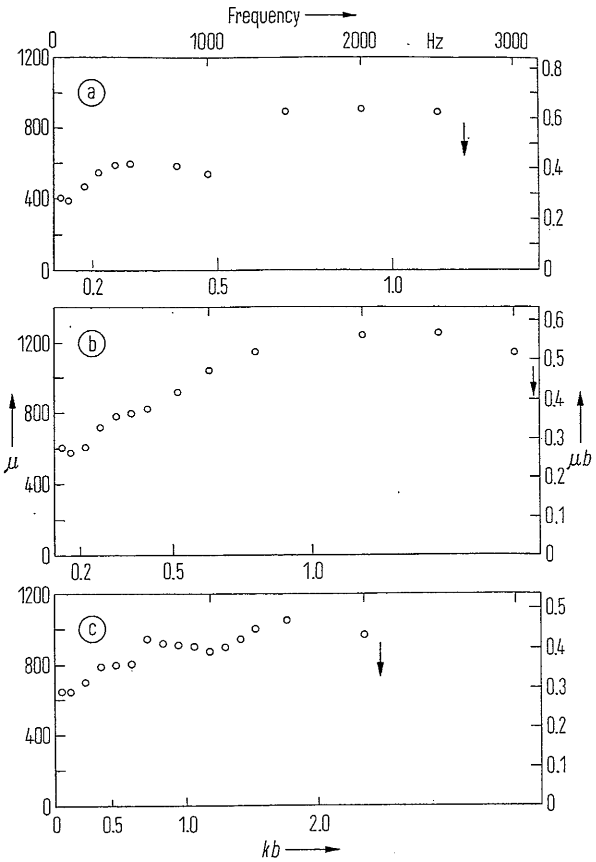


Fig. 9. Measured values, as a function of frequency ($\omega/2\pi$), of the average amplification factor μ along a jet of length 8 mm, with average width parameter b and blowing pressure p_0 for
(a) slit width 1 mm, $b \approx 0.7$ mm, $p_0 = 100$ Pa,
(b) slit width 0.5 mm, $b \approx 0.45$ mm, $p_0 = 80$ Pa,
(c) slit width 0.5 mm, $b \approx 0.45$ mm, $p_0 = 40$ Pa.
Note that the scale for kb ($k = \omega/u$) is non-linear.

then rise to a value between about 0.5 and 0.7 which is maintained approximately until the sharp drop to zero at the high frequency limit.

In summary then, we conclude that, over the range of jet parameters studied, the inviscid theory is essentially correct in its prediction of the amplification coefficient for frequencies low enough that $\mu b \leqq 0.6$ and in its prediction that no amplification occurs above about $\mu b \approx 2$. In the range $0.6 < \mu b < 2$, however, the experimental results differ from the theory.

8. Turbulent jets

Our measurements and analyses have been confined to jets with sufficiently low Reynolds numbers that, under ordinary laboratory conditions, they are almost certainly non-turbulent. We have, however, identified what appears to be a change in behaviour when the Reynolds number exceeds about 1000, and many jets of interest in musical instruments lie in this higher velocity regime.

Unfortunately the noise components of the jet turbulence interfere with the behaviour of our measurement system in its present form so that no results can be presented. From a simple theoretical viewpoint, provided the scale of the turbulence remains small compared with that of the jet flow, we might speculate that its major effect may be to increase the effective kinematic viscosity ν by an amount depending on the Reynolds number of the flow, so that

$$\nu = \nu_0[1 + f(Vb/\nu_0)] \tag{39}$$

where ν_0 is the kinematic viscosity in the absence of eddy diffusion. If the function f is dominated by its linear term then substitution of eq. (39) in eq. (34) suggests that, when eddy diffusion is dominant,

$$c_\infty \to \text{const} \times V. \tag{40}$$

It will be instructive to test this speculation experimentally.

9. Conclusions

Our experiments have shown that the propagation behaviour of sinuous waves on real air jets is rather different from the behaviour predicted by the best of available modern theories. The classic treatment given by Rayleigh for inviscid jets still serves as the fundamental scheme in terms of which jet behaviour can be understood but the existence of a non-zero fluid viscosity apparently has effects in addition to those related to its modification of the jet velocity profile.

Measurements in the laminar regime show that wave velocity is essentially constant along the length of the divergent jet, its magnitude being related not to the jet velocity V but rather to the integral J of V^2 across the width of the jet. For small values of the angular frequency ω the wave velocity varies as $(J\omega)^{1/3}$ as suggested by the theory, while for larger ω the wave velocity becomes independent of frequency and simply proportional to J. Behaviour in the turbulent regime has not yet been studied.

The amplification factor μ also behaves essentially as predicted by inviscid theory for low frequencies and exhibits a cut-off at close to the frequency predicted theoretically. In the upper half of this frequency range, however, experiment shows a nearly constant value of μ while the theory predicts a smooth decrease towards zero.

Solution of the Navier-Stokes equations to provide a theoretical treatment of this behaviour is beyond our resources. We have, however, developed a very much simplified discussion based on dimensional analysis which appears to provide a reasonably satisfactory first-order description of the observations and which may serve as a guide in the formulation of a more detailed theory.

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