Nonlinearity, Complexity, and the Sounds of Musical Instruments

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Abstract: Sustained-tone musical instruments can be classified as "essentially nonlinear" because their musical operation depends critically upon a major nonlinearity in the sound generating mechanism. Impulsively-excited instruments, on the other hand, are "incidentally nonlinear" despite the fact that nonlinearity sometimes contributes a great deal to their sound. These apparent paradoxes are explained in detail, with the aid of examples.

INTRODUCTION

Most elementary accounts of the acoustics of musical instruments treat them as simple, linear, harmonic systems. The natural modes of a taut string, of a cylindrical tube, and of a conical horn are, after all, in simple harmonic relationship, and measurements of the spectra of musical instruments are often quoted as showing that their overtones are harmonics of the fundamental "to better than one part in ten thousand." The one exception in this exposition is usually the bell, for which the overtones are clearly not harmonically related, and this is accounted for by the peculiar thick-walled profile.

While such descriptions are quite satisfactory at their intended level, a closer examination of the physics involved shows that the real situation is, however, vastly different. The overtones in the sound of a bowed-string instrument are indeed precise phase-locked harmonics of the fundamental, as can be seen from the unchanging waveform in mechanically bowed instruments, despite the fact that the stiffness of real strings makes their natural modes significantly inharmonic. The same is true of wind instruments, in which the end-corrections at the mouth and throat vary significantly with frequency to give appreciably inharmonic modes, while finger holes can lead to very non-integral mode frequency ratios. For such instruments to work at all, in a musical sense, a high level of nonlinearity is essential, as is shown below, so that they may reasonably be classified as "essentially nonlinear."

For impulsively excited instruments, such as pianos, guitars, and percussion instruments, on the other hand, nonlinearity is weaker and can be classed as "incidental" — the instrument would work even if its behavior were strictly linear. Nevertheless, the presence of nonlinearity makes a major contribution to the complexity of the sounds of instruments such as gongs and cymbals.

This paper investigates this apparent paradox and highlights the role of nonlinearity in sound generation in both sustained-tone and impulsively-excited musical instruments. A detailed analytical treatment of musical instrument acoustics is available elsewhere (1).

SUSTAINED-TONE INSTRUMENTS

In a sustained-tone instrument, a resonator that is substantially linear in its properties, though not necessarily harmonic, is closely coupled to a generator that is itself driven by a steady force — bow motion or air pressure — and controlled by the resulting oscillations so as to contribute a negative mechanical or acoustic resistance. If this generator were strictly linear, then it would set a large number of resonator modes into oscillation at frequencies little different from their natural frequencies, as determined by the reactive part of the generator impedance, and the sound would be entirely different from the phase-locked harmonic sounds to which we are accustomed. This possibility, and its consequences for music, has been explored by Sethares (2) using computer-generated sounds, but will not concern us here.

In reality, the generator in a sustained-tone musical instrument is always highly nonlinear in its response, as shown in the examples of Figure 1. In each case there is a negative resistance region, but the magnitude of this resistance varies over each cycle of the resonator oscillation. In the case of a bowed string and a vibrating reed there is a sharp additional nonlinearity at one end of the operating range. As the mathematics in the Appendix shows, this nonlinearity can lead to shifts in the operating frequencies of all the modes until they are locked into precise harmonic relationship, giving the familiar sounds and spectra of Western musical instruments.

This fails only when the modes are extremely inharmonic in their frequency relationships, as for example with peculiar non-standard fingerings on woodwind instruments, and the generator is manipulated so as to reduce its nonlinearity. Under these circumstances, harmonic frequency locking may be simply impossible, and each mode will oscillate at a frequency close to its natural resonance. The nonlinearity will generally be sufficient, however, to generate multiple sum and difference tones with frequencies $N\omega_n \pm M\omega_m$, where N and M are small integers. From the mathematics, the amplitude of this difference tone will decrease markedly as the integers N and M increase. The resultant peculiar sound is generally called a multiphonic and is much loved by avant garde composers. Under special conditions, the behavior may be even more complex than this and degenerate into subharmonic or even apparently chaotic sounds (3).

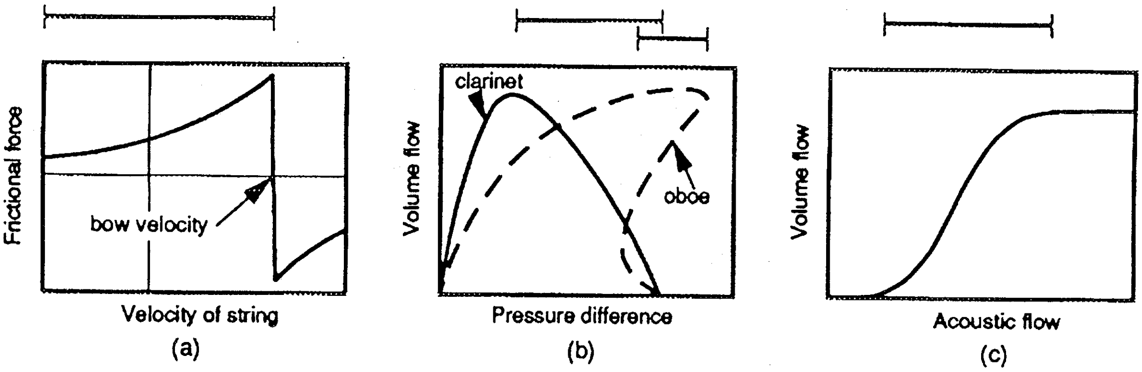

FIGURE 1. Generator characteristics of (a) a moving bow exciting a string, (b) the air flow through a vibrating reed, and (c) the air jet in a flute. The negative resistance region is indicated in each case.

IMPULSIVELY EXCITED INSTRUMENTS

Impulsively excited instruments can be classified as "incidentally nonlinear" because, at small excitation amplitudes, their sound output is based entirely upon their natural mode frequencies, whether these be harmoncially related or not. Plucked or hammered string instruments have nearly harmonic mode frequencies in any case, with $\omega_n = n\omega_1(1+\gamma n^2)$, where the inharmonicity constant γ depends upon the ratio of string stiffness (diameter and material) to string tension, and is generally much less than unity in musical instruments. Nonlinear effects enter when the vibration amplitude becomes large enough to increase the string tension significantly. The pitch of all modes then rises, though not to exactly the same extent. This effect is heard as an unpleasant descending "twang" as the amplitude decays after excitation. Both inharmonicity and pitch glide are minimized by high tensile stress, and thus, in instruments such as grand pianos, by long string scaling. Membrane drums behave in much the same way as strings in relation to pitch-glide effects.

Plucked strings exhibit one other interesting and important effect that arises from coupling between the two polarization directions of the string vibration through the nonlinear tension (4). This coupling causes the plane of polarization to rotate, typically at a rate of order 1 revolution per second for the fundamental, and this appears as a slow variation in the sound output, because the two polarizations directions are generally differently coupled to the soundboard of the instrument.

In thick-shell idiophones the normal restoring forces are provided not by tension but rather by material stiffness, which varies as the third power of the material thickness. Since nonlinear tension forces vary only linearly with material thickness, there is thus a progression in nonlinear behavior from bells to gongs to cymbals as the material becomes thinner. This is illustrated in Figure 2. Bells, with their very thick shells, are essentially linear in behavior, except for generation of the second harmonic of the fundamental because this mode distorts the normal circular plan section to an ellipse, which has smaller area, and thus expells some of the contained air at twice the fundamental frequency. The warbling effect noted in some bells is simply a consequence of slight mistuning between pairs of nominally degenerate modes, caused by slight asymmetries, rather than by any nonlinear effect.

Very similar conclusions apply to the heavy gongs of the Indonesian Gamelan orchestra, and also to percussion instruments such as the triangle, the glockenspiel and the marimba, which have quite solid vibrating elements.

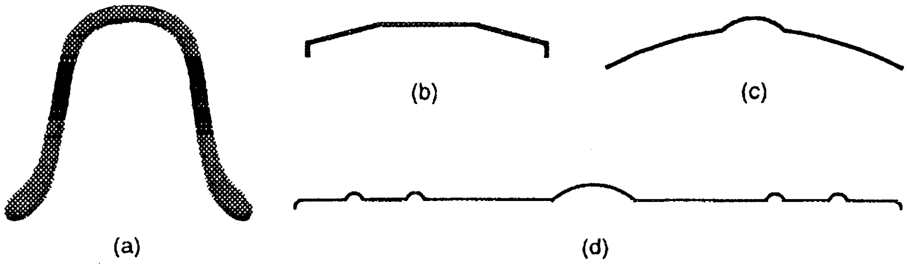

FIGURE 2. Profiles of (a) a typical bell, (b) a Chinese Opera gong, (c) a cymbal and (d) a tamtam. There is a progression in nonlinearity along this series.

The first interesting case is that of the Chinese Opera gong, shown in Figure 2(b). The vibrating section is in the center, while the conical surround simply serves to stiffen it and baffle the radiation. A soft stroke with a padded stick simply excites the fundamental mode, but interesting nonlinear effect emerge in loud playing. In the first version of this gong, the central section is flat, so that excitation of any normal mode gives rise to a tension stress that is proportional to the square of the mode displacement x, just as in a string, giving a restoring force of the form $\alpha x + \gamma x^3$. Essentially only the fundamental mode is excited when the gong is struck centrally with a padded stick, and the pitch of this glides downwards by as much as a musical third after a vigorous stroke. In a second version of this gong, the central portion is very slightly domed, leading to a restoring force of the form $\alpha x + \beta x^2 + \gamma x^3$. Analysis (5) shows that, for reasonably accessible amplitudes, the pitch is actually depressed by as much as a musical third, and then glides upwards, rather than downwards, as the vibration decays. Together these two gongs create a memorable effect in Peking Opera performances.

The cymbal, shown in Figure 2(c) is relatively stiff because of its slightly conical shape, but its characteristic sound suggests something more than a simple superposition of its normal modes. An experiment to demonstrate this, which works almost equally well with a simple flat plate, involves exciting the cymbal sinusoidally with a shaker attached to its central boss. If the excitation frequency is chosen to match approximately the fundamental frequency, typically about 600 Hz, and the excitation amplitude is progressively raised, then the sound initially consists of a superposition of this fundamental and an increasing intensity of its harmonics, as shown in Figure 3(a). At a critical amplitude, however, one of two things can happen, depending upon the exact frequency.

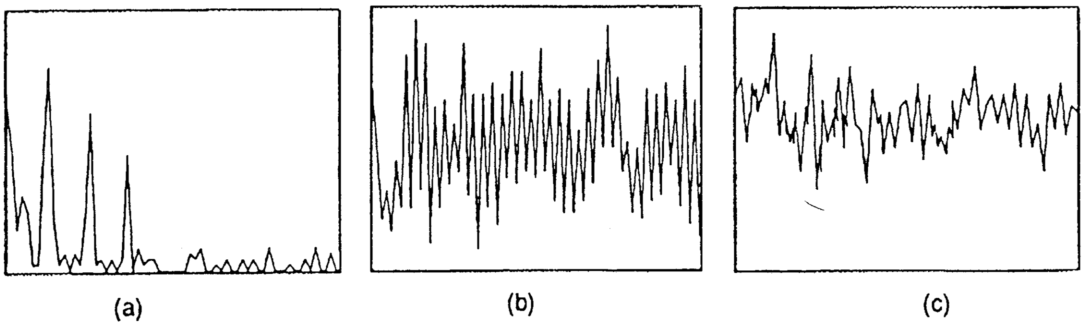

FIGURE 3. Radiated sound spectrum for a cymbal sinusoidally excited at its center: (a) excitation below the critical bifurcation level; (b) complex regime with subharmonic of order 5; (c) apparently chaotic vibration.

In the first case, the behavior undergoes a bifurcation and generates a subharmonic of order 2, 3 or 5, together with all its harmonics, in the cases studied. The subharmonic 5 case is particularly effective aurally. Before the bifurcation, the note is a rather strident E_5 in the treble cleff, which is followed by a rich C-major chord based upon C_2, a little more than two octaves lower in pitch, as shown in Figure 3(b).

The other pattern of behavior is perhaps more relevant to the sound of a struck cymbal. For excitation at about the same level but at a slightly different frequency, the cymbal sound makes an abrupt transition to what appears to be a chaotic vibration, the sound of which is very similar to that of a struck cymbal, as shown in Figure 3(c). An exact analysis is difficult because the cymbal has an infinite number of degrees of freedom, but some exploratory studies have been carried out (6).

Finally we come to the case of the Chinese tamtam of Figure 2(c). This large gong, about a meter in diameter, is only a millimeter or so thick and nearly flat except for a shallow circumferential ring, a central dome and several rings of hammered bumps. These changes in slope appear to be important in the explanation of the behavior of this gong. Once again, tension forces, which are important because of the relative thinness of the gong shell, vary at frequency 2ω, if ω is the frequency of the mode involved. At an abrupt change in slope, this tension force combines with the original mode displacement to generate a transverse force of frequency 3ω. The rings of 100 or so evenly spaced bumps couple these higher-frequency forcing terms to shell modes with similar angular variation, which are necessarily localized around the periphery of the gong. Thus the energy of the low-frequency (about 80 Hz) centrally symmetric mode, initially excited by a blow from a large padded hammer, is gradually transferred to high frequency modes (6). The localization of these modes at the periphery is easily checked experimentally.

This upward cascade of energy to higher modes occurs over a time of a second or more, as shown in Figure 4. The higher modes are closely spaced in frequency in any case, but it is possible that there is also a transition to chaotic behavior as in the cymbal. The large tamtam, used sparingly, gives an immensely impressive sound in the modern symphony orchestra.

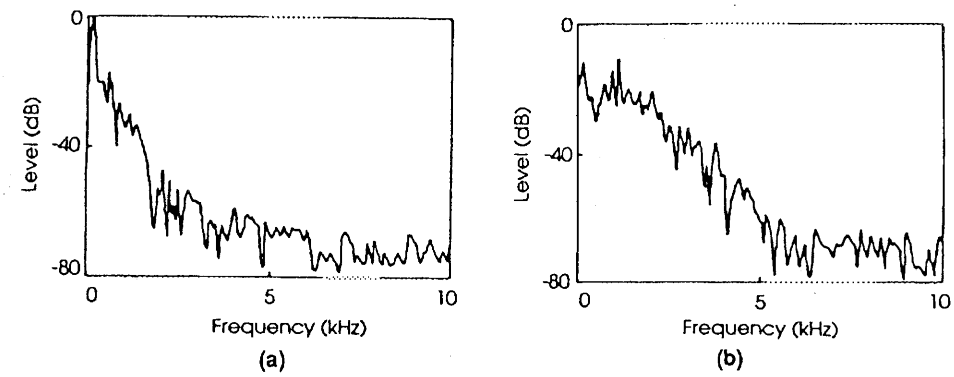

FIGURE 4. Radiated sound spectrum of a large tamtam (a) immediately after excitation by a central blow from a large padded hammer, and (b) after a further 3 seconds. Note the transfer of energy from low frequency to high frequency modes.

CONCLUSION

This brief survey shows that nonlinearity is very important in accounting for the characteristic sounds of musical instruments. The sounds of sustained-tone instruments, conditioned to harmonicity by strong nonlinear effects, have been responsible for the development of Western music, and its distinction from much Eastern music that is conditioned by the nonharmonic sounds of percussion idiophones. Studies of nonlinearity are now a vital part of musical acoustics, and indeed have implications for related nonlinear treatments of the human voice and of birdsong (7,8).

APPENDIX: A LITTLE MATHEMATICS

Suppose that the acoustic quantity associated with the resonator is $x = \Sigma_n x_n$, where $x_n = a_n \sin(\omega_n t + \phi_n)$ is the excitation of the nth mode of natural frequency ω_n. Suppose also that the amplitudes a_n and phases ϕ_n are slowly varying functions of time. Then it can be shown (1) that, if we retain only slowly varying terms as indicated by the notation $[\ldots]_0$, then

$$da_n/dt = (1/\omega_n)\,[F(x)\cos(\omega_n t + \phi_n)]_0 - \alpha_n a_n \quad (1)$$

$$d\phi_n/dt = -(1/a_n\omega_n)\,[F(x)\sin(\omega_n t + \phi_n)]_0 \quad (2)$$

where α is the damping parameter and $F(x)$ includes both the forcing function supplied by the generator and also any nonlinear coupling between the modes. F is in general a nonlinear function of x, and therefore of all the modes x_n.

The interesting thing about these equations is that the actual vibration frequency for mode n is $\omega_n + d\phi_n/dt$ which can be very different from ω_n if the nonlinearity and amplitude of the fundamental are both large enough. In the case of a continuously excited system, if the nonlinearity is great enough to move all the modes to the nearest harmonically related frequencies, the system will settle down to a steady state with $da_n/dt = 0$ for all n. These well defined mode amplitudes will be phase-locked into harmonic frequency relationships. For impulsive excitation, on the other hand, there may initially be large shifts in mode frequencies if the mode interactions are nonlinear, but the vibration will slowly decay towards simple superposed modes, since nonlinear terms decrease more rapidly than linear terms.

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