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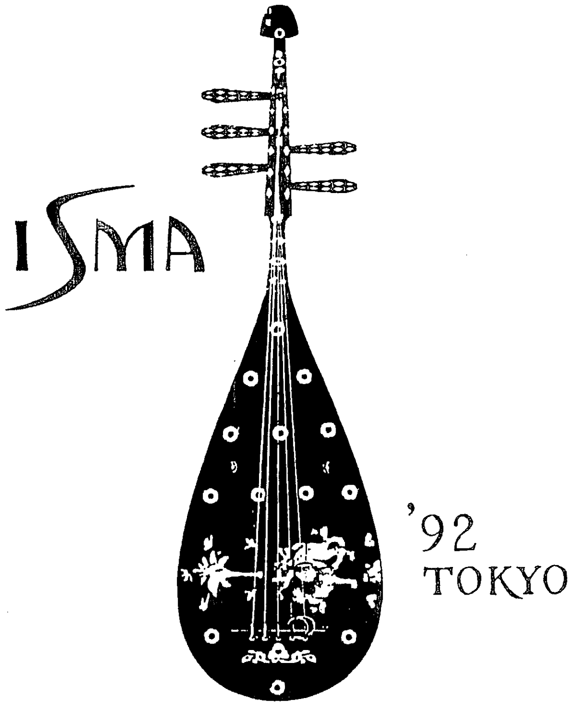

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LINEARITY AND NONLINEARITY IN MUSICAL INSTRUMENTS

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Abstract: Basic understanding of musical instruments relies upon the fact that the primary vibrating element is, in nearly all cases, a simple linear extended vibrator with well defined mode frequencies. Some instruments, such as bells, guitars, and harpsichords, can be almost completely understood in this approximation, though the discussion may necessarily be quite complicated because of the number of coupled linear systems involved. Instruments producing sustained sound, however, such as violins, clarinets, and trumpets, depend essentially for their operation upon nonlinear phenomena. These nonlinear phenomena largely determine the radiated acoustic power, the frequency spectrum, and the transient behaviour fo the instrument. There has been great advance in our understanding of these matters over the past twenty years, and this paper will give an overview of musical instrument behaviour from this point of view.

1. LINEAR SYSTEMS

Musical instruments such as guitars, harpsichords, and bells have in common the fact that they are impulsively excited nearly linear systems. (By "linear" we mean that doubling the amplitude of the exciting force simply doubles the amplitude of each partial in the radiated acoustic spectrum.) The primary vibrating element—the string in the case of guitars and harpsichords and the whole shell in the case of bells—possesses an infinite number of normal modes, and these are excited to a greater or less extent, depending on the location and strength of the impulse and the nature of the exciting mechanism. The elastic stiffness of the vibrating element, and the non-zero compliance of the deflecting or striking mechanism, combine to ensure that there is a roll-off slope in the excitation of the vibrating element of at least 12 dB/octave at high frequencies. The frequency or mode number above which this roll-off applies varies greatly with the circumstances.

In the case of bells, the vibrating body is itself also the radiating element, so that the radiated sound closely follows the pattern of the mechanical vibration. The mode frequencies are determined by the exact shape and thickness distribution of the bell body, and are generally tuned to approximate simple musical intervals, though not a simple harmonic series, in order to give a pleasant sound. The rotational symmetry of an ordinary bell dictates a two-fold degeneracy in all the modes, but geometrical imperfections generally break this symmetry so that the mode pairs produce audible slow beats.

In guitars, harpsichords, and other plucked-string instruments, the vibrating string is usually thin and flexible, so that its normal modes constitute a nearly perfect harmonic series. Only when the strings are rather thicker and made from metal does their mode series exhibit slightly stretched upper partials and produce a slightly bell-like sound, as in the upper strings of the piano. The strings themselves are, however, not the radiating elements, instead they must excite a rather large soundboard which then radiates the sound. This feature has two consequences. In the first place, it takes time for energy to be transferred from the vibrating string to the soundboard, and both the transfer and the subsequent radiation efficiency depend upon the mechanical impedance of the soundboard at the excitation point. This shapes not only the attack transient, but also the overall radiated spectrum. In the second place, since the instrument may have many strings, some of them tuned as unison groups, the simple degeneracy between vertical and horizontal vibration is not only broken by the differing impedance of the bridge and soundboard in these two directions, but also influenced by coupling between strings through the bridge. A complete description of the operation of such an instrument may therefore be very complex, but it is still linear in all respects. Only when the initial deflection of the string becomes large enough that its tension is significantly increased do we find a trace of nonlinear behaviour, clearly audible as a descending twang at the beginning of the note. This generally undesirable situation is avoided by using gut or nylon strings with relatively low Young's modulus, and by making the string tension as high as reasonably possible.

Two exceptions to this discussion are worthy of note. The first is the Indian sitar, in which one of the bridges is very gently curved and almost tangent to the string lying across it. String vibration in a plane normal to the instrument face then causes the contact point of the string on the bridge, and thus the vibrating length of the string, to vary periodically. The amplitude of this length variation is related to the amplitude of the normal string vibration. This behaviour is nonlinear because both variations enter the equation of motion of the string. The result is the typical and unusual tone of the instrument, which does not appear to have been analysed in detail.

The second exception is the piano, in which not the string but rather the hammer excitation mechanism is significantly nonlinear. The felt in a piano hammer is graded in density within the hammer, with denser felt in the interior, so that the effective elastic compliance of the hammer is reduced when the exciting stroke is vigorous. This, in turn, reduces the length of time the hammer spends in contact with the string, and so affects the tone, a vigorous keystroke giving a brighter as well as louder sound. This has been analysed in detail by Hall.

2. NONLINEAR IMPULSIVELY-EXCITED INSTRUMENTS

In contrast to the linearly-behaved instruments discussed above are the shallow percussion instruments of the gong and cymbal family, which owe much of their characteristic sound quality to extreme nonlinearity. There is, of course, a gradual transition from bells to gongs as the depth of the curved shell is reduced and the wall thickness decreased. Nonlinearity first becomes apparent in instruments such as Chinese opera gongs, which have a nearly flat vibrating plate surrounded by a thicker conical flange with a turned-down edge to give further stiffness. If the central section of the gong is quite flat, then the elastic restoring force is provided by the nearly linear elastic stiffness of the plate material, supplemented by a tensional restoring force that increases as the square of the mode amplitude. A vigorous blow on such a gong therefore leads to a sound in which the frequency glides downwards as the amplitude decays. A pitch glide of about 20%, corresponding to a major third, can be achieved. A companion gong has a central section that is slightly domed, and both analysis and experiment show that in this case the tensional forces tend initially to reduce the mode frequencies as mode amplitude is increased, before raising them for very large amplitudes. In an actual gong of this type, only the upward-gliding pitch behaviour is accessible, and again the glide amounts to as much as 20% for a vigorous blow.

For large and very shallow gongs of the Chinese tamtam family, the effects produced by vibrational nonlinearity are much more extreme. For a gentle blow with a large and softly padded beater, the gong behaves nearly linearly and radiates its characteristic mode frequencies. Increasingly vigorous blows excite harmonics of the basic modes and then, for still harder blows, the sound quality suddenly changes to a rich blaze of shimmering sound. Measurement shows that the vibrational energy, initially communicated to just a few of the lowest centrosymmetric modes, is transferred progressively to modes of higher frequency, and particularly to those having angular symmetry related to the angular symmetry of the rings of bumps hammered into the surface of the gong. Still closer analysis shows that the vibration of the gong no longer consists of a simple superposition of natural modes but is actually chaotic, in the technical sense of the term. Cymbals are also found to have chaotic behaviour, though the transition is generally masked by the fact that it is traditional to excite them either by a glancing blow from another cymbal or by a sharp blow on the periphery with a hard stick. Both these methods excite predominantly the high-frequency modes with high angular dependence, which already have a dense spectrum rather like that developed during chaotic vibration.

Detailed analysis of this transition to chaos is complicated by the high dimensionality of the problem—the number of modes involved is potentially very large. A beginning has been made, but we are still very far from a complete understanding of the phenomena.

3. SUSTAINED-TONE INSTRUMENTS

Instruments with sustained tone are generally based on strings or air columns as vibrating elements. A simple analysis suggests that this is because these can be made to have a nearly harmonically related set of overtones, even for peculiar bore shapes such as found in brass instruments, but more careful examination shows that the situation is, in reality, much more complex than this. Just as the partials of real strings are not in exact harmonic relation, so too, and in a much more extreme way, the upper modes of cylindrical pipes and complex horns differ quite markedly from such a simple series because of frequency-dependent end corrections, open finger holes, and other subtleties. The normal sounds produced by all these instruments, however, consist of partials that are in exact and phase-locked harmonic relationship, as can be easily seen from the unchanging shape of the radiated sound waveform. The spectrum of the sound output, whether from a violin, a flute, or a trumpet, is characteristic of the instrument, the playing level, and, to some extent, the player. Studies of the nonlinear behaviour of sustained-tone wind instruments were begun long ago by Benade, and of course such studies for bowed strings have an even longer history.

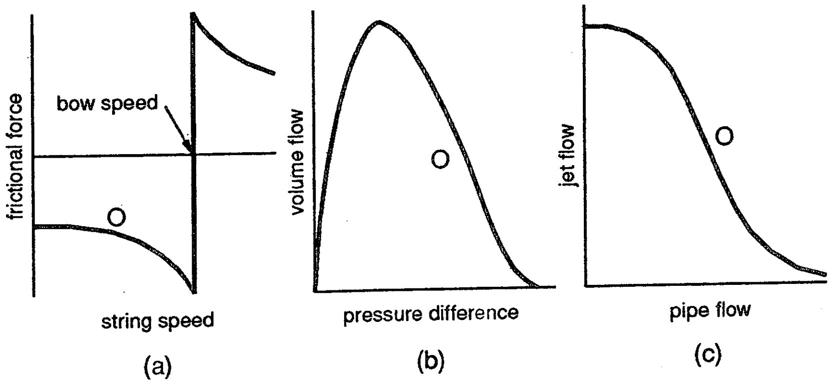

Figure 1. The negative resistance characteristics of (a) a bowed string, (b) a blown reed, and (c) an air jet. The operating point is set near O in each case. Note that the characteristic becomes nonlinear, and the negative-resistance slope decreases, for large excursions away from O.

The mechanism by which a steady mechanical or pneumatic power source—the motion of a frictional bow, or the steady air supply from the lungs or a mechanical bellows—is converted to a oscillatory motion can be understood, initially, as a simple positive feedback process which becomes linearised in the small-signal limit as some sort of negative resistance. In the case of a bowed string, this negative resistance arises from the variation of frictional force with slipping speed; in the case of an air column driven by a reed or lip generator, it comes from the behaviour of flow through an aperture whose size depends on the pressure drop across it; in the case of an air column driven by an air jet, the negative resistance has its origin in the phase delay for transverse waves propagating along the jet. These three mechanisms are illustrated in Fig. 1.

The problem with a linear mechanism, however, is that it excites all modes of the vibrator, though perhaps with some frequency selectivity associated with driving point or phase, and these modes then oscillate with their own characteristic, and generally inharmonic, natural frequencies. Even more notable than this, a linear mechanism provides no clue to limitation of the vibration amplitude, though clearly some nonlinear effect must intervene when the power available from the generator is exceeded.

In fact, there is very significant nonlinearity in all three of the generators we have described: the frictional force exerted by the bow on the string changes sign abruptly when the relative velocity passes through zero; the acoustic conductance of a reed generator changes sign at one end of the oscillatory swing and clamps to zero at the other when the reed closes; and the acoustic conductance of an air-jet generator falls to zero at both ends of the oscillatory excursion when it blows either entirely into or entirely outside the pipe. With the exception of bowed friction, these changes are not abrupt, and the nonlinearity increases progressively as the amplitude of the acoustic vibration increases.

It is possible to analyse the effects of such progressive nonlinearity quite generally, as well as relating it to the behaviour of particular generator systems. Essentially what we do is to calculate, either in the time domain or the frequency domain, the behaviour of the nonlinear generator driving the linear multimode resonant system which is the instrument itself, and then finally examine the acoustic signal radiated by the extended system, since acoustic radiation is generally a small perturbation on top of other kinds of damping. Three general conclusions are reached.

- Provided that some of the principal modes of a system driven by a nonlinear generator have frequencies that are reasonably close to being related as the ratio of small integers, it is possible to generate a stable oscillation in which these modes are locked into precisely integral frequency relation and constant relative phase to produce a precisely repetitive waveform and an exactly harmonic spectrum. The oscillation of modes not fitting this spectrum is suppressed. Achievement of this oscillation regime is facilitated if the generator nonlinearity is large and the mismatch of mode frequencies is small. The relative levels of the harmonics are determined by the nature of the nonlinearity and the distribution of modes of the instrument. Again there is a roll-off behaviour of at least 12 dB/octave above some characteristic cut-off frequency.
- The oscillation builds up slowly, generally over some tens of cycles, and its ultimate amplitude is limited by system dissipation and by available power from the steady source. Details are determined by system nonlinearity and by acoustic impedance relationships. The overall efficiency with which steady power is converted to radiated acoustic power rarely exceeds 1%, and can be much less than this figure, particularly near the excitation threshold for sound production.
- Musical instruments have been evolved to play in this harmonic regime. It is possible, however, by using unusual fingerings (in the case of woodwind instruments), or unusual playing techniques (in relation to bow speed or pressure, blowing pressure, or lip configuration) to produce situations in which mode frequencies are significantly displaced and nonlinearity is reduced. In such cases the system may be able to sustain oscillations based on two or more different modes simultaneously. The nonlinearity of the generator then ensures that multiple sum and difference frequencies are present, and the result is an unusual, but often musically useful, "multiphonic" sound.

The behaviour of many musical instruments has been analysed very fully in the normal playing regime, often without particular thought being given to the necessity for justifying the existence of such a regime. More recently, increasing attention has been given to multiphonics and other nontraditional instrumental sounds. Indeed recent analysis by Kergomard, Keefe, and others suggests that some multiphonic sounds may actually be chaotic, rather than simply multiply periodic as suggested above.