Transfer matrix of conical waveguides with any geometric parameters for increased precision in computer modeling

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Abstract: The existing formula for the transfer matrix of conical elements assumes constant wave number, which is only valid for sufficiently short conical elements. In acoustic waveguides, the phase velocity, attenuation constant, and hence complex wave number depend on frequency and cross-section radius. As for conical waveguides, the cross-section radius is position dependent, the transfer matrix must allow for a position-dependent wave number. Taking this into account, this letter presents an analytic derivation of the transfer matrix for conical waveguides with any geometric parameters, which can be utilized to improve the method of computer modeling of complex waveguides.

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1. Introduction

The pressure and volume flow at one end of a waveguide are related to the pressure and volume flow at the other end by a transfer matrix.[1–3] This transfer matrix also determines the relation between input impedance and load impedance. Acoustic waveguides are modeled as a series of cylindrical and conical elements. Such analyses have applications in computer modeling of air-conditioning ducts,[4] exhaust mufflers,[5] and wind instruments in musical acoustics.[6,7] The phase velocity v and attenuation constant α, and hence the complex wave number k, depend on the cross-section radius R as well as on the frequency f. As the radius of the cross section is constant throughout a cylindrical element, k is constant, and the transfer matrix is given by Olson[2] and Fletcher and Rossing.[3] An expression for the transfer matrix of a conical element, neglecting the dependence on cross-section radius and assuming constant complex wave number, is given by Olson[2] and Fletcher and Rossing.[3] This letter presents a derivation of the transfer matrix of a conical element, of any length, any inlet and outlet radii, including the dependence of the complex wave number k on the cross-section radius R, where for a conical element, the cross-section radius is not constant but a function of position, in contrast to the case of a cylindrical element.

As a result of viscous and thermal wall losses, which increase as the radius is decreased, the complex wave number k in an acoustic waveguide is a function of cross-section radius R and for propagation of sound waves in air for $r_v > 10$ and to a good approximation for $r_v > 3$, this dependence is given[1,3,8] by the following relation:

$$k = \frac{2\pi}{\lambda} - i\alpha = \frac{2\pi f}{v(R)} - i\alpha(R) = \frac{2\pi f}{c}\left(1 - \frac{\delta}{Rf^{1/2}}\right)^{-1} - i\frac{\varepsilon f^{1/2}}{R}, \tag{1}$$

where $\delta = 1.65 \times 10^{-3}$ m Hz$^{1/2}$ and $\varepsilon = 3 \times 10^{-5}$ s$^{1/2}$. (The nondimensional parameter $r_v = R/(\eta/\omega\rho)^{1/2}$ is the ratio of the waveguide's cross-section radius to the viscous boundary layer thickness, which is equal to the square root of the dynamic viscosity η divided by the product of the angular frequency ω and density ρ.)

The existing computational method for modeling complex waveguides using transfer matrices consists of dividing them into cylindrical and conical elements, which make up the

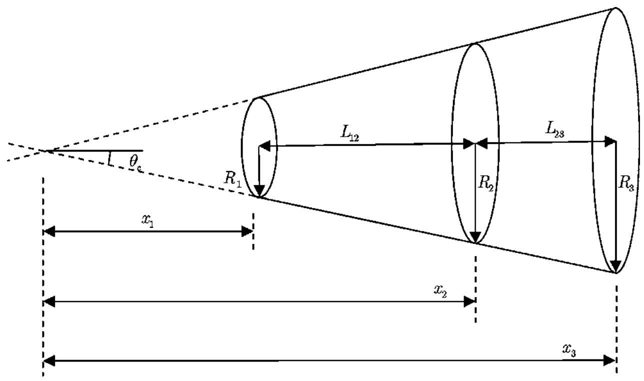

Fig. 1. Two successive conical elements of the same cone angle, from the same truncated cone with the geometric parameters of each conical element shown.

entire waveguide and for each frequency, numerically multiplying the transfer matrices to determine the transfer matrix of the whole complex waveguide. For computational methods of modeling waveguides and ducts, the reader is referred to.[9–11] For applications of computational methods for modeling wind instruments in musical acoustics, refer to.[3,6,7,12–14] Applications of numerical techniques in the design of mufflers in exhaust systems are discussed in Refs. 15 and 16.

The formula derived in this letter, for the transfer matrix of a conical waveguide has a convenient algebraic form and can be used to improve the method of computer modeling of acoustic systems, including air-conditioning ducts, exhaust mufflers and wind instruments in musical acoustics, increasing precision and reducing computation time. Using this formula, the transfer matrix for a long conical waveguide or a long conical component of a waveguide can be calculated accurately without the need to split it into short conical elements and multiply through a large number of transfer matrices numerically for each frequency, to achieve accurate results.

2. Analytic derivation

For a conical element of infinitesimal length, the complex wave number is constant. A finite conical waveguide may be analyzed as a set of infinitesimal conical elements. Fig. 1 shows two successive conical elements with the same cone angle.

Let the first conical element be between cross sections at distances x_1 and x_2 from the cone apex with length $L_{12}=x_2-x_1$ and effective wave number k_{12}. The transfer matrix for the first conical element may be written (in accordance with Ref. 3) as:

$$M_{12}=\frac{x_2}{x_1}\begin{pmatrix} -t_2\sin(k_{12}L_{12}-\theta_2) & i\sin(k_{12}L_{12}) \\ it_1t_2\sin(k_{12}L_{12}-\theta_2+\theta_1) & t_1\sin(k_{12}L_{12}+\theta_1)\end{pmatrix}$$
$$=\frac{x_2}{x_1}\begin{pmatrix} t_2\cos\left(k_{12}L_{12}-\left(\theta_2-\frac{\pi}{2}\right)\right) & i\sin(k_{12}L_{12}) \\ it_1t_2\sin(k_{12}L_{12}-\theta_2+\theta_1) & t_1\cos\left(k_{12}L_{12}+\left(\theta_1-\frac{\pi}{2}\right)\right)\end{pmatrix}. \quad (2)$$

In addition to those parameters already mentioned, the transfer matrix depends on several parameters characterizing the two ends of the conical element: $\theta_1=\arctan(k_1x_1)$, $\theta_2=\arctan(k_2x_2)$, $t_1=1/\sin\theta_1$ and $t_2=1/\sin\theta_2$.

The transfer matrix links the pressure P_1 and volume flow U_1 at one end of the conical element to the pressure P_2 and volume flow U_2 at the other end, via the following relation:

$$\begin{pmatrix} P_1 \\ Z_1 U_1 \end{pmatrix} = M_{12} \begin{pmatrix} P_2 \\ Z_2 U_2 \end{pmatrix}. \tag{3}$$

In this relation, $Z_1 = \rho c / S_1$ and $Z_2 = \rho c / S_2$ are the characteristic impedances at each of the two ends of the conical element, where S_1 and S_2 are the respective cross-sectional areas at each of the two ends, ρ is the density of the medium, c is the speed of sound in the medium. Analogous relations to Eqs. (2) and (3) define the transfer matrix of the second conical element.

The transfer matrix M_{13} for the combination of two conical elements is found by matrix multiplication of the two transfer matrices M_{12} and M_{23} of the two conical elements. By using trigonometric identities for all four entries of this resulting matrix, the transfer matrix for the combination of 2 conical elements simplifies to the following form:

$$\begin{aligned} M_{13} &= M_{12} M_{23} \\ &= \frac{x_3}{x_1} \begin{pmatrix} -t_3 \sin(k_{12}L_{12} + k_{23}L_{23} - \theta_3) & i \sin(k_{12}L_{12} + k_{23}L_{23}) \\ i t_1 t_3 \sin(k_{12}L_{12} + k_{23}L_{23} + \theta_1 - \theta_3) & t_1 \sin(k_{12}L_{12} + k_{23}L_{23} + \theta_1) \end{pmatrix} \\ &= \frac{x_3}{x_1} \begin{pmatrix} -t_3 \sin(\zeta_{12} + \zeta_{23} - \theta_3) & i \sin(\zeta_{12} + \zeta_{23}) \\ i t_1 t_3 \sin(\zeta_{12} + \zeta_{23} + \theta_1 - \theta_3) & t_1 \sin(\zeta_{12} + \zeta_{23} + \theta_1) \end{pmatrix} \\ &= \frac{x_3}{x_1} \begin{pmatrix} -t_3 \sin(\zeta_{13} - \theta_3) & i \sin(\zeta_{13}) \\ i t_1 t_3 \sin(\zeta_{13} + \theta_1 - \theta_3) & t_1 \sin(\zeta_{13} + \theta_1) \end{pmatrix}. \end{aligned} \tag{4}$$

This has the same form as the matrix for a single conical element with $\zeta = kL$ replaced by $\zeta_{13} = \zeta_{12} + \zeta_{23} = k_{12}L_{12} + k_{23}L_{23}$, where $\zeta_{ij} = k_{ij}L_{ij}$ may be considered the non-dimensional length of the conical element, since it is an additive parameter for successive conical elements.

For a conical waveguide, the cross-section radius R depends on the position x relative to the cone apex. In order to allow for the dependence of the complex wave number k on the cross-section radius R, and therefore on the position x relative to the cone apex, it is necessary to first divide the conical waveguide into infinitesimal elements.

To this end, Eq. (4) is extended to N conical elements, by mathematical induction:

$$\begin{aligned} M_{0N} &= M_{01} M_{12} M_{23} M_{34} \cdots M_{N-1,N} \\ &= \frac{x_N}{x_0} \begin{pmatrix} -t_N \sin(\zeta_{01} + \zeta_{12} + \cdots + \zeta_{N-1,N} - \theta_N) & i \sin(\zeta_{01} + \zeta_{12} + \cdots + \zeta_{N-1,N}) \\ i t_0 t_N \sin(\zeta_{01} + \zeta_{12} + \cdots + \zeta_{N-1,N} + \theta_0 - \theta_N) & t_0 \sin(\zeta_{01} + \zeta_{12} + \cdots + \zeta_{N-1,N} + \theta_0) \end{pmatrix} \\ &= \frac{x_N}{x_0} \begin{pmatrix} -t_N \sin(\zeta_{0N} - \theta_N) & i \sin(\zeta_{0N}) \\ i t_0 t_N \sin(\zeta_{0N} + \theta_0 - \theta_N) & t_0 \sin(\zeta_{0N} + \theta_0) \end{pmatrix}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \zeta_{0N} &= \zeta_{01} + \zeta_{12} + \zeta_{23} + \zeta_{34} + \cdots + \zeta_{(N-1),N} \\ &= k_{01}L_{01} + k_{12}L_{12} + k_{23}L_{23} + k_{34}L_{34} + \cdots + k_{(N-1),N}L_{(N-1),N} \\ &= \sum_{i=0}^{N-1} k_{i,i+1} \Delta x_i, \end{aligned} \tag{6}$$

and where

$$\Delta x_i = L_{i,i+1} = x_{i+1} - x_i. \tag{7}$$

In the limit as N goes to infinity and $\Delta x_i \to 0$, while keeping fixed the endpoints of the conical waveguide and its length $L = L_{0,N} = \Sigma_{i=0}^{N-1} \Delta x_i$, the Riemann sum in Eq. (6) above becomes the integral

$$\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx = L\left(\frac{1}{L}\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx\right) = \bar{k}L, \tag{8}$$

where $L = x_{\text{out}} - x_{\text{in}}$ is the length of the finite conical waveguide and $\bar{k} := (1/L)\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)\,dx$ is the effective wave number of the finite conical waveguide.

Therefore, the transfer matrix for a finite conical waveguide with any geometric parameters is

$$M = \frac{x_{\text{out}}}{x_{\text{in}}}\begin{pmatrix} -t_{\text{out}}\sin\left(\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx - \theta_{\text{out}}\right) & i\sin\left(\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx\right) \\ it_{\text{in}}t_{\text{out}}\sin\left(\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx + \theta_{\text{in}} - \theta_{\text{out}}\right) & t_{\text{in}}\sin\left(\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx + \theta_{\text{in}}\right)\end{pmatrix}$$
$$= \frac{x_{\text{out}}}{x_{\text{in}}}\begin{pmatrix} -t_{\text{out}}\sin(\bar{k}L - \theta_{\text{out}}) & i\sin(\bar{k}L) \\ it_{\text{in}}t_{\text{out}}\sin(\bar{k}L + \theta_{\text{in}} - \theta_{\text{out}}) & t_{\text{in}}\sin(\bar{k}L + \theta_{\text{in}})\end{pmatrix}. \tag{9}$$

This formula for the transfer matrix of a conical waveguide of any geometrical parameters is the main result of this letter.

An explicit expression can be written for the transfer matrix, including the spatial dependence of the complex wave number arising from the position-dependent cross-section radius in a conical waveguide, by expressing k in terms of x using Eq. (1) and the relation $R = x\tan(\theta_c)$, where θ_c is the half-angle of the cone. Thus

$$k(x) = \frac{2\pi f}{c}\left(1 - \frac{\delta}{\tan(\theta_c)xf^{1/2}}\right)^{-1} - i\frac{\varepsilon f^{1/2}}{\tan(\theta_c)x}. \tag{10}$$

After some rearrangement, this gives

$$k(x) = \frac{2\pi f}{c}\left(1 + \frac{b}{x-b}\right) - i\frac{\varepsilon f^{1/2}}{\tan(\theta_c)}\frac{1}{x},$$

where $b = \delta/\tan(\theta_c)f^{1/2}$. This expression may be integrated to give

$$\int_{x_{\text{in}}}^{x_{\text{out}}} k(x)dx = \frac{2\pi f}{c}\left(\int_{x_{\text{in}}}^{x_{\text{out}}} 1dx + \int_{x_{\text{in}}}^{x_{\text{out}}} \frac{b}{x-b}dx\right) - i\frac{\varepsilon f^{1/2}}{\tan(\theta_c)}\int_{x_{\text{in}}}^{x_{\text{out}}} \frac{1}{x}dx$$
$$= \frac{2\pi f}{c}\left\{L + b\ln\left(\frac{x_{\text{out}} - b}{x_{\text{in}} - b}\right)\right\} - i\frac{\varepsilon f^{1/2}}{\tan(\theta_c)}\ln\left(\frac{x_{\text{out}}}{x_{\text{in}}}\right)$$
$$= \frac{2\pi f}{c}\left\{L + \frac{\delta}{\tan(\theta_c)f^{1/2}}\ln\left(\frac{x_{\text{out}} - \dfrac{\delta}{\tan(\theta_c)f^{1/2}}}{x_{\text{in}} - \dfrac{\delta}{\tan(\theta_c)f^{1/2}}}\right)\right\} - i\frac{\varepsilon f^{1/2}}{\tan(\theta_c)}\ln\left(\frac{x_{\text{out}}}{x_{\text{in}}}\right) \tag{11}$$

where $\delta = 1.65\times10^{-3}$ m Hz$^{1/2}$ and $\varepsilon = 3\times10^{-5}$ s$^{1/2}$. Substitution of Eq. (11) into Eq. (9) gives an algebraic expression for the transfer matrix of a finite conical waveguide with any geometric parameters.

3. Special case of sufficiently small conical elements

For a short conical element ($|L| \ll x_{\text{in}}$), the best one-term Taylor expansion approximation for $\ln(x_{\text{out}}/x_{\text{in}})$ in terms of the small parameter L/x^* is at $x^* = \bar{x} = (x_{\text{in}} + x_{\text{out}})/2$, where the error for

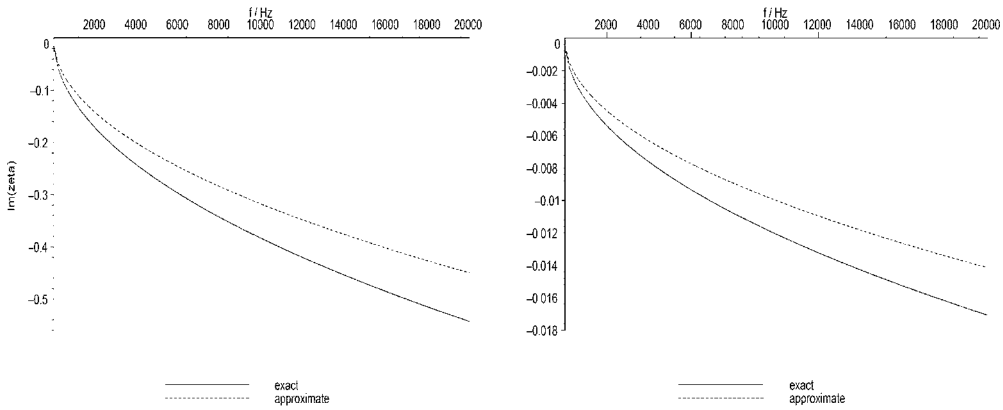

Fig. 2. Imaginary part of the approximate expression $\zeta_{\text{approx}}=k(\bar{x})L$ and the exact integral expression $\zeta_{\text{exact}}=\int_{x_{\text{in}}}^{x_{\text{out}}}k(x)dx$ for a conical waveguide with inlet radius 1 mm and outlet radius 5 mm having a length of (a) 318 mm and (b) 10 mm.

this approximation is third order and not second order as for an approximation at any other x^* between x_{in} and x_{out}. Consider the logarithm in the real part of the integral. The Taylor approximation requires the stronger condition that $|L| \ll |x^*-b|$, i.e. $|L| \ll |x^*-\delta/\tan(\theta_c)f^{1/2}|$. Even for a small conical element with $|L| \ll x^*$, the approximation may be invalid for sufficiently small frequencies or sufficiently small cone angles.

The approximation is a consistent over-approximation by $O((L/2(\bar{x}-b))^3)$ at $x^*=\bar{x}=(x_{\text{in}}+x_{\text{out}})/2$ and a consistent under-approximation or over-approximation by $O((L/(x^*-b))^2)$ for any other x^* between x_{in} and x_{out}. Thus, the best possible approximation retaining only the first term of the Taylor expansion is at $x^*=\bar{x}=(x_{\text{in}}+x_{\text{out}})/2$ with an error of order $O((L/2(\bar{x}-b))^3)$. Thus, for sufficiently small conical elements the best one-term Taylor approximation of the form $k(x^*)L$ for the integral expression is $k(\bar{x})L$ at the midpoint of the conical element.

Figure 2 shows for comparison, the imaginary part of the approximate expression $\zeta_{\text{approx}}=k(\bar{x})L$ and the exact integral expression $\zeta_{\text{exact}}=\int_{x_{\text{in}}}^{x_{\text{out}}}k(x)dx$ for particular values of the geometric parameters of a conical waveguide. Each expression is shown as a function of frequency and the difference between the two curves is clearly significant particularly for long and narrow waveguides.

4. Conclusion

The effective wave number of a conical waveguide cannot be determined only from the wave numbers at the two ends. Thus:

$$\bar{k} \neq \frac{k(x_{\text{in}})+k(x_{\text{out}})}{2}, \quad \bar{k} \neq \sqrt{k(x_{\text{in}})k(x_{\text{out}})}, \quad \bar{k} \neq \{[k(x_{\text{in}})]^{-1}+[k(x_{\text{out}})]^{-1}\}^{-1}.$$

The effective wave number depends not only on the wave number at the endpoints or at the midpoint of the waveguide, but in fact on the wave number at all points throughout the waveguide.

In this letter, the transfer matrix of a conical waveguide has been analytically derived, taking into account that the complex wave number depends on the cross-section radius, which is a function of position for a conical waveguide. As a result the complex wave number is a function of position in a conical waveguide, rather than a constant throughout its length as is the case for a cylindrical waveguide. Thus, the derived transfer matrix of a conical waveguide is valid for any length, inlet and outlet radii, having taken into account the spatial dependence of the wave number. This is of particular interest in the computer modeling of acoustic systems, where high precision is important. This includes computer modeling of air-conditioning ducts,[4] exhaust mufflers,[5] and wind instruments in musical acoustics.[6,7]

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