A/ Prof. C. Ulrich

PHYSICS PHYS1121 – HIGHER PHYSICS 1131

4. OSCILLATIONS AND WAVES

4.2 Waves

4.2.1 Travelling Waves

- Waves are periodic oscillations in space and time.
- A wave transports energy but not matter.
- Mechanical waves require a medium.
- Electromagnetic waves (e.g. light waves) do not require a medium.

A Pulse

- A pulse is not periodic, but travels also in space and time.
- A pulse transports energy but not mass.
- A pulse has a certain height (amplitude).
- A pulse does not change much as it travels.
- Its motion can be
  - transversal (e.g. water waves)
  - longitudinal (e.g. sound waves)

A pulse travelling along a string:

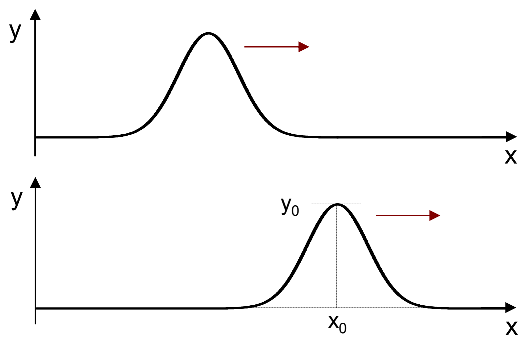

Figure 4.9: A pulse travelling along a string.

$$y(x_0, 0) = f(x_0) \qquad \text{height of the pulse}$$

After a certain time the pulse has travelled a distance of $x = v \cdot t$.

The new position is:

$$x' = x_0 + v \cdot t$$
$$y(x', t) = f(x + vt)$$

Actually, for a pulse travelling to the right side we should use the minus sign for the correct mathematical sense: $y(x', t) = f(x - vt)$.

Travelling Waves

The simplest form to describe a travelling wave is a sinusodial function:

$$\begin{aligned} y(x,t) &= A \cdot \sin\left(\tfrac{2\pi}{\lambda}(x - vt) + \phi\right) \\ &= A \cdot \sin\left((kx - \omega t) + \phi\right) \end{aligned}$$

where $\vec{k} = \frac{2\pi}{\lambda}$ is the wavevector and $\omega = \frac{2\pi}{T}$ is the angular frequency. The wavevector $\vec{k}$ is pointing in the propagation direction of the wave.

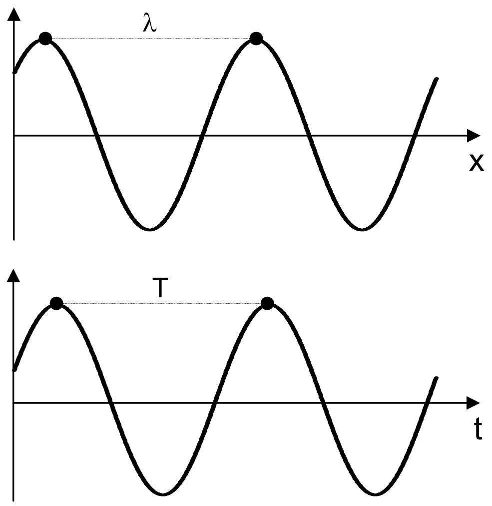

Figure 4.10: A wave is travelling is space and time and has a wavelength of λ (wavevector: $\vec{k} = \frac{2\pi}{\lambda}$) and a period of T (angular frequency $\omega = \frac{2\pi}{T}$).

The wave travels with a velocity of v and has travelled a distance of $v \cdot t$ after a certain time t:

$$x' = x + v \cdot t$$

After one wavelength λ the wave repeats itself. The function which fulfills this criterion is:

$$y(x,t) = A \cdot \sin\left(\tfrac{2\pi}{\lambda}(x - vt) + \phi\right)$$

In each period of 2π the wave has travelled a distance of λ in the time of one period T.

Note that the velocity is:

$$v = \frac{\lambda}{T} = f \cdot \lambda$$

$$y(x,t) = A \cdot \sin\left(\tfrac{2\pi}{\lambda}x - 2\pi f t) + \phi\right)$$

$$\boxed{y(x,t) = A \cdot \sin(kx - \omega t + \phi)}$$

The phase-shift ϕ is determined by the starting point (in space and time) of the wave.

Relation between a simple harmonic motion and a wave

A simple harmonic motion acts like a source which generates a wave. The wave is then travelling away from this source

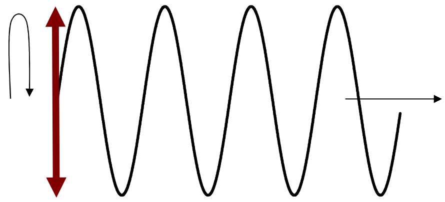

Figure 4.11: A simple harmonic motion can generate a travelling wave.

4.2.2 The Velocity of a Wave which is travelling along a String

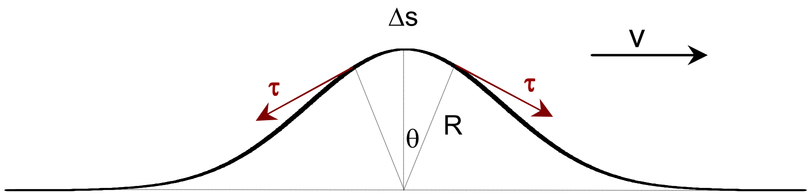

Figure 4.12: Tension on a string caused by a travelling wave.

$$\Delta s = 2\,R\,\tan\theta = 2\,R\,\theta \qquad \text{for small angles: } \sin\theta \approx \tan\theta \approx \theta$$

The tension τ is the only restoring force.

On the other side, a mass element on the string gets accelerated in up- and down-direction: $F = m\,a_y$.

With the centripetal acceleration $a_y = \frac{v^2}{R}$ the centripetal force is:

$$\frac{m\,v^2}{R} = 2\,\tau\,\sin\theta = 2\,\tau\,\theta$$

The moving mass element of a section Δs on the string is $m = \mu \cdot \Delta s$, where μ is the linear mass density.

Therefore the following expression can be derived:

$$m = \mu \cdot \Delta s = \mu \cdot 2\,R\,\theta$$

$$2\,\tau\,\theta = \frac{\mu \cdot 2\,R\,\theta\,v^2}{R} \qquad \text{and therefore: } \tau = \mu\,v^2$$

$$\boxed{v = \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{\text{elastic properties}}{\text{inertial properties}}}}$$

Example

A string has a length of $6\,m$ and a mass of $0.30\,kg$. Attached to the string is a mass of $2\,kg$, which creates a tension on the string. What is the velocity of a travelling wave on this string?

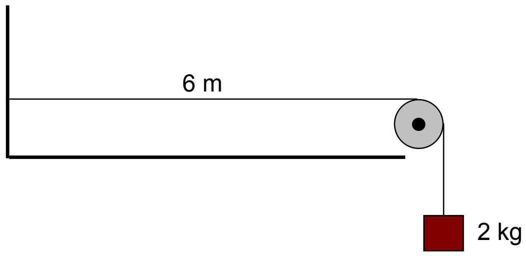

Figure 4.13: Example: velocity of a travelling wave on a string under tension.

$$\begin{aligned} \text{linear mass density} \quad \mu &= \frac{0.30\,kg}{6\,m} = 0.05\,\frac{kg}{m} \\ \text{tension} \quad \tau &= 2\,kg \cdot 10\,\frac{m}{s^2} = 20\,N \\ \text{velocity} \quad v &= \sqrt{\frac{\tau}{\mu}} = \sqrt{\frac{20\,N}{0.05\,kg/m}} = 20\,\frac{m}{s} \end{aligned}$$

4.2.3 Energy and Power of a Travelling Wave

The kinetic energy is:

$$K = \frac{1}{2}\,m\,v^2$$

The kinetic energy of a certain mass element on the string ($d\,m$) is:

$$d\,K = \frac{1}{2}\,(d\,m)\,v^2$$

Using the wavefunction: $y(x,t) = A \cdot \sin(kx - \omega t + \phi)$, one obtains the transversal speed $u(t)$ for the up and down movement of the mass:

$$u(t) = \frac{d\,y(x,t)}{d\,t} = -\,\omega\,A \cdot \cos(kx - \omega t + \phi)$$

The kinetic energy of the mass element is then:

$$\begin{aligned} d\,K &= \frac{1}{2}\,(d\,m)\,\omega^2\,A^2 \cdot \cos^2(kx - \omega t + \phi) \\ &= \frac{1}{2}\,\mu\,dx\,\omega^2\,A^2 \cdot \cos^2(kx - \omega t + \phi) \end{aligned}$$

with the linear mass density μ.
Divide both sides of the equation be $d\,t$ and insert the velocity of the travelling wave: $v = \frac{\lambda}{T} = \frac{d\,x}{d\,t}$:

$$\frac{d\,K}{d\,t} = \frac{1}{2}\,\mu \cdot v \cdot \omega^2\,A^2 \cdot \cos^2(kx - \omega t + \phi)$$

Taking the average gives the following expression:

$$\left\langle \frac{d\,K}{d\,t} \right\rangle_{\text{avg.}} = \frac{1}{2}\,\mu\cdot v\cdot\omega^2\,A^2\cdot\langle\cos^2(kx-\omega t+\phi)\rangle_{\text{avg.}}$$
$$= \frac{1}{4}\,\mu\cdot v\cdot\omega^2\,A^2$$

The average power is:

$$P_{\text{avg.}} = 2\cdot\left(\frac{d\,K}{d\,t}\right)_{\text{avg.}} = \frac{1}{2}\,\mu\cdot v\cdot\omega^2\cdot A^2$$

The factor 2 is coming from the fact that $E_{\text{total}} = E_{\text{kin}} + E_{\text{pot}} = E_{\text{kin,max.}} = 2\cdot E_{\text{kin,avg.}}$.

4.2.4 The Wave Equation

$$\boxed{\frac{d^2y(x,t)}{dx^2} = \frac{1}{v^2}\,\frac{d^2y(x,t)}{dt^2}}$$

In order to derive this equation we consider a small mass element of the string. This mass moves up and down, i.e. it is constantly accelerated up and down in vertical direction:

$$a_y = \frac{d^2y(x,t)}{dt^2}$$

And the mass element is $m = \mu\cdot d\,x$.
On the other side the string experiences a tension τ along the horizontal direction:

$$F = \sqrt{F_x^2+F_y^2} = \tau$$

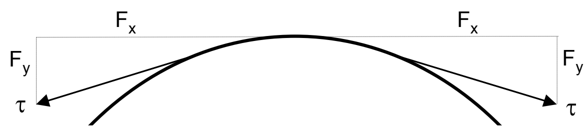

Figure 4.14: Forces in x- and y-direction on a string when a wave travels along the string.

s is the slope of the force: $s = \frac{F_y}{F_x}$.

By assuming that $F_x \gg F_y$ one can use the approximation that F_x is the tension: $F_x = \tau$.

$$F_y = F_x \cdot s = \tau \cdot s$$

$$\begin{aligned}
\tau\, s_1 - \tau\, s_2 &= \mu \cdot d\,x \,\frac{d^2\, y}{d\,t^2} \\
\frac{s_2 - s_2}{d\,x} &= \frac{\mu}{\tau}\,\frac{d^2\, y}{d\,t^2} \\
\frac{d\,s}{d\,x} &= \frac{\mu}{\tau}\,\frac{d^2\, y}{d\,t^2} \qquad \text{where} \quad s = \frac{dy}{dx} \\
\frac{d^2\, y}{d\,x^2} &= \frac{\mu}{\tau}\,\frac{d^2\, y}{d\,t^2} \qquad \text{note :} \quad v = \sqrt{\frac{\tau}{\mu}} \\
\frac{d^2\, y}{d\,x^2} &= \frac{1}{v^2}\,\frac{d^2\, y}{d\,t^2}
\end{aligned}$$

This is the equation of motion of a travelling wave, or the **wave equation**!

Wave Equation:

$$\boxed{\frac{d^2 y(x,t)}{dx^2} - \frac{1}{v^2}\,\frac{d^2 y(x,t)}{dt^2} = 0}$$

The wave equation is a second order differential equation in space and time.

In order to solve this equation we assume that the following wave function might be a solution of the wave equation:

$$\boxed{y(x,t) = A \cdot \sin(kx - \omega t + \phi)}$$

$$\begin{aligned}
\frac{d\,y(x,t)}{d\,x} &= A\,k \cdot \cos(kx - \omega t + \phi) \\
\frac{d^2\,y(x,t)}{d\,x^2} &= -\,A\,k^2 \cdot \sin(kx - \omega t + \phi) \\
\frac{d\,y(x,t)}{d\,t} &= -\,A\,\omega \cdot \cos(kx - \omega t + \phi) \\
\frac{d^2\,y(x,t)}{d\,t^2} &= -\,A\,\omega^2 \cdot \sin(kx - \omega t + \phi)
\end{aligned}$$

Inserted into the wave equation gives:

$$\begin{aligned}
-\,A\,k^2 \cdot \sin(kx - \omega t + \phi) &= \frac{1}{v^2}\,\big(-\,A\,\omega^2 \cdot \sin(kx - \omega t + \phi)\big) \\
-\,k^2 &= -\,\frac{1}{v^2}\,\omega^2 \qquad \text{and} \qquad v = \frac{\omega}{k} = \frac{\lambda}{T}.
\end{aligned}$$

This demonstrates that our above given wavefunction is a solution of the wave equation.

In principle every sine- or cosine-function with the same period T and wavelength λ is a possible solution of this wave equation.

4.2.5 Superposition of Waves

Principle of Superposition

Any linear combination of sine- or cosine-solutions of the wave equation is again a solution of the wave equation.

$$\Psi(x,t) \quad = \quad y_1(x,t) \ + \ y_1(x,t) \ = \ A_1 \cdot \sin(kx - \omega t + \phi_1) \ + \ A_2 \cdot \cos(kx - \omega t + \phi_2)$$

Note: Identities for Trigonometric Functions

$$\begin{aligned} \sin A \ + \sin B &= 2 \sin\left(\frac{A \pm B}{2}\right) \cdot \cos\left(\frac{A \mp B}{2}\right) \\ \cos A \ + \cos B &= 2 \cos\left(\frac{A + B}{2}\right) \cdot \cos\left(\frac{A - B}{2}\right) \\ \cos A \ - \cos B &= -2 \sin\left(\frac{A + B}{2}\right) \cdot \sin\left(\frac{A - B}{2}\right) \end{aligned}$$

Example:

$$\begin{aligned} y_1(x,t) &= A \cdot \sin(kx - \omega t) \\ y_2(x,t) &= A \cdot \sin(kx - \omega t + \phi) \end{aligned}$$

$$\begin{aligned} y_1(x,t) \ + \ y_1(x,t) &= A \ (\sin(kx - \omega t) \ + \ \sin(kx - \omega t + \phi)) \\ &= 2\, A \cdot \sin\left(kx - \omega t + \frac{\phi}{2}\right) \cdot \cos\left(\frac{\phi}{2}\right) \end{aligned}$$

How does the solution look like?

Case I: constructive interference

$$\phi \quad = \quad 0 \qquad \longrightarrow \qquad \cos\left(\frac{\phi}{2}\right) = 1$$

Constructive interference is obtained if the phase shift between both waves is $s = n \cdot \lambda$.

Case II: destructive interference

$$\phi \quad = \quad \pi \qquad \longrightarrow \qquad \cos\left(\frac{\phi}{2}\right) = 0$$

A phase-shift of $\phi = \pi$ corresponds to a shift of the wave by $T/2$ or $\lambda/2$.

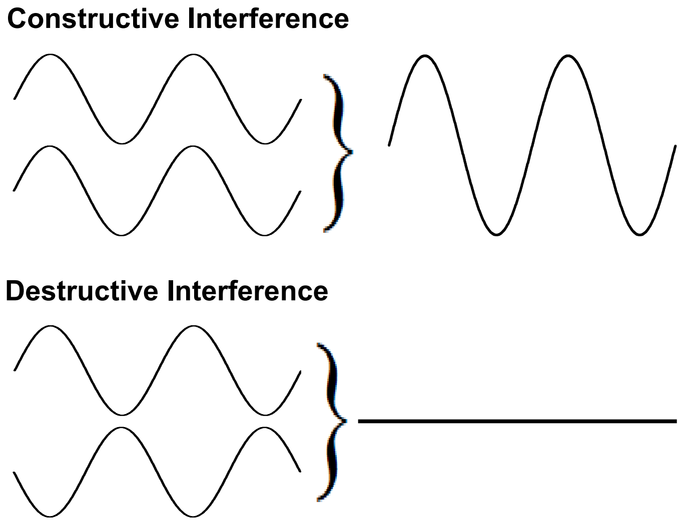

Figure 4.15: Constructive and destructive interference of two waves with a phase shift of $\phi = 0$ and $\phi = \pi$, respectively.

Case III: arbitrary phase shift

Let us chose a phase-shift of $\phi = \pi/3$ for example:

$$\phi \quad = \quad \pi/3 \qquad \longrightarrow \qquad \cos\left(\frac{\phi}{2}\right) = \frac{\sqrt{3}}{2}$$

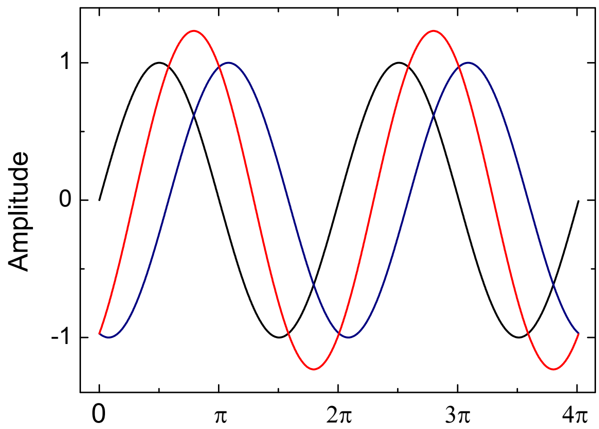

Figure 4.16: Superposition of two waves with a phase-shift of $\phi = \pi/3$. The red line is the final resulting wave.

4.2.6 Reflection, Transmission and Superposition of Waves

Reflection of a Pulse

- When a pulse gets reflected from a fixed end, it gets reverted.

  The support fixes the string and causes a force in the opposite direction.
- When a pulse is reflected on a loose end, the pulse is not reverted.

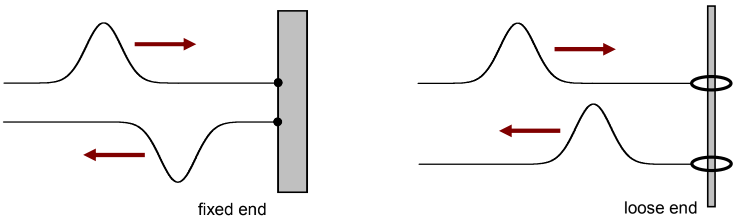

Figure 4.17: Reflection of a pulse on a fixed and a loose end.

When a pulse reaches a junction of two ropes with different linear mass density, a part of the pulse is transmitted and a part of the pulse is reflected.

The velocity of the pulses of each part, i.e. the reflected and transmitted has the velocity of $v = \sqrt{\frac{\tau}{\mu}}$.

The pulse travels faster in the less dense medium (lower linear mass density).

Passing Pulses

Two pulses pass through each other without being destroyed or altered.

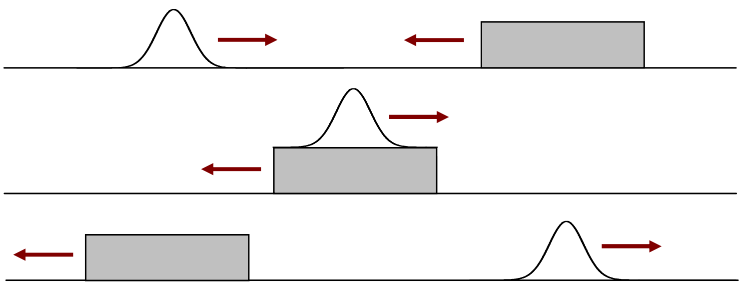

Figure 4.18: Two pulses are passing each other without being destroyed or altered.

4.2.7 Doppler Effect

1. The observer moves towards the source:

An observer moved towards the source:

- The frequency f and the wavelength λ of the wave emitted from the source do not change.

- However, the frequency measured by the observer does change.

$$v' = v_{\text{sound}} + v_{\text{observer}}$$

where $v_{\text{sound}} = v$ is the velocity of the sound wave emitted from the source and $v_{\text{observer}} = v_o$ is the velocity of the observer.

$$f' = \frac{v'}{\lambda} = \frac{v + v_o}{\lambda} \quad \text{and} \quad v = \lambda \cdot f$$

$$f' = \left(\frac{v + v_o}{v}\right) \cdot f$$

2. The observer moves away from the source:

$$v' = v_{\text{sound}} - v_{\text{observer}}$$

$$f' = \frac{v - v_o}{\lambda} = \left(\frac{v - v_o}{v}\right) \cdot f$$

3. The observer is at rest but the source moves towards the observer:

The wavelength is decreased. During one oscillation the source has travelled a distance of $s = \Delta\lambda = v_{\text{Source}} \cdot T = v_s \cdot T$:

$$\lambda' = \lambda - \Delta\lambda = \lambda - v_s \cdot T = \lambda - \frac{v_s}{f}$$

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda - (v_s/f)} = \frac{v}{v/f - v_s/f} = \left(\frac{v}{v - v_s}\right) \cdot f$$

4. The observer is at rest but the source moves away from the observer:

$$f' = \left(\frac{v}{v + v_s}\right) \cdot f$$

5. Both, source and observer move towards each other:

$$f' = \left(\frac{v + v_o}{v - v_s}\right) \cdot f$$

6. Both, source and observer move away from each other:

$$f' = \left(\frac{v - v_o}{v + v_s}\right) \cdot f$$