A/ Prof. C. Ulrich

PHYSICS PHYS1121 – HIGHER PHYSICS 1131

4. OSCILLATIONS AND WAVES

4.1 Harmonic Oscillation

4.1.1 Springs and the Simple Harmonic Motion (SHM)

Periodic Motion

A periodic motion is a motion of an object that regularly returns to a given position after a fixed time interval.

Simple Harmonic Motion (SHM)

- The force is proportional to the distance from the equilibrium position.

- The force is directed towards the equilibrium position.

- The potential energy is a quadratic function with respect to the distance.

Hooke's Law

$$F_{\text{Spring}} = -k\,x$$

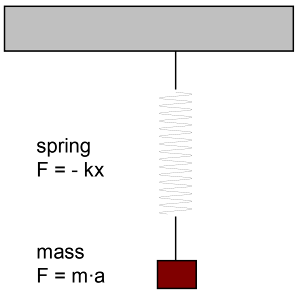

Figure 4.1: Harmonic Oscillation of a mass at a spring.

- At the maximum elongation the spring is pulling on the mass.
- The mass gets accelerated towards the equilibrium position.
- At the equilibrium position the acceleration is zero and the velocity of the mass reaches its maximum.
- After passing through the equilibrium position the movement of the mass slows down. The acceleration is negative until the mass stops and then returns.
- The strength and direction of the acceleration changes all the time.

Newton's Law

$$F_{\text{Mass}} \quad = \quad m \cdot a$$

Due to Newton's law **'actio = re-actio'** both forces must be equal at all time during the motion of the mass: $F_{\text{Spring}} = F_{\text{Mass}}$.

$$-k\,x(t) \quad = \quad m \cdot a(t)$$

$$m \cdot a(t) + k\,x(t) \quad = \quad 0$$

$$a(t) \quad = \quad -\frac{k}{m}\,x(t)$$

- The acceleration is maximum when the displacement is at its maximum since at this point the force from the spring is maximum.
- At this point the potential energy of the spring reaches its maximum $E_{\text{Spring}} = \frac{1}{2}kx^2$.
- At the equilibrium position the acceleration is zero since the force of the spring is zero.
- However, at the equilibrium position the speed of the mass reaches its maximum and the kinetic energy of the mass is at its maximum with $E_{\text{kin}} = \frac{1}{2}mv^2_{\text{max.}}$.

The acceleration is:

$$a(t) \quad = \quad \frac{d\,v(t)}{dt} = \frac{d^2x(t)}{dt^2}$$

Equation of Motion

$$m\,\frac{d^2x(t)}{dt^2} \quad = \quad -\,k\,x(t)$$

$$m\,\frac{d^2x(t)}{dt^2} \; + \; k\,x(t) \quad = \quad 0$$

This is the equation of motion of the mass attached to a spring. The movement is an oscillation of the mass around the equilibrium position.

The equation of motion is a second order differential equation in time.

How would a possible solution of this equation look like?

What do we know so far? It is an oscillation!

Therefore we can assume that it might be a sine- or cosine-function:

$$x(t) = A \cdot \cos(\omega t + \phi)$$

where A is the amplitude, i.e. the maximum elongation from the equilibrium position.

ϕ is a phase shift and is defined by the starting point of the observation.

$\omega = \frac{2\pi}{T}$ is the angular frequency, where T is the duration (period) of one oscillation.

Let's try if this function for $x(t)$ is a possible solution of the equation of motion:

$$\begin{aligned} x(t) &= A \cdot \cos(\omega t + \phi) \\ \frac{dx(t)}{dt} &= -A\,\omega \cdot \sin(\omega t + \phi) \\ \frac{d^2x(t)}{dt^2} &= -A\,\omega^2 \cdot \cos(\omega t + \phi) \end{aligned}$$

Insert into the equation of motion:

$$-m \cdot A\,\omega^2 \cdot \cos(\omega t + \phi) + k \cdot A \cdot \cos(\omega t + \phi) = 0$$

$$-m\,\omega^2 + k = 0$$

$$\omega^2 = \frac{k}{m} \qquad \omega = \sqrt{\frac{k}{m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \qquad f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The equation of motion has two solutions:

1. Yes, our assumption is correct:
   $x(t) = A\,\cos(\omega t + \phi)$ is a possible solution of the wave equation.
2. The angular frequency is $\omega = \sqrt{\frac{k}{m}}$.

Some considerations about the oscillation:

At the maximum of $x(t)$ the following identity holds:

$$\begin{aligned} \cos(\omega\,t + \phi) &= 1 \\ \omega\,t + \phi &= 0 \\ \omega\,t &= -\phi \quad \text{and therefore} \quad \phi = -\frac{2\pi}{T} \cdot t \end{aligned}$$

where t is the starting time of the observation of the oscillation.

1^{st} maximum	$\omega t + \phi = 0$
2^{nd} maximum	$\omega(t + T) + \phi = 2\pi$

$$\text{therefore:} \quad \omega\,T = 2\pi \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

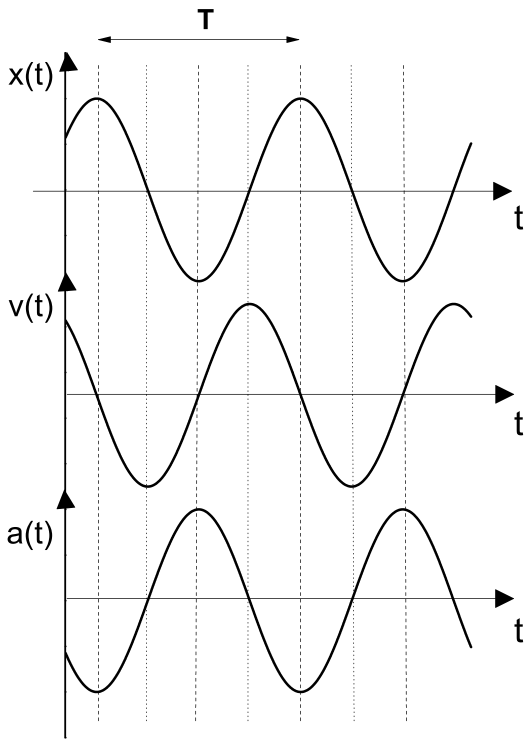

Figure 4.2: Position, velocity, and acceleration of a mass during a harmonic oscillation around an equilibrium position as a function of time.

The **period** T and the **frequency** f of the oscillation are:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

$$T = 2\pi\sqrt{\frac{m}{k}}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Energy of the Oscillation

The total energy of the system must remain constant (**conservation of energy**).

- **Potential energy**: the spring $E_{\text{pot.}} = \frac{1}{2}\,k\,x(t)^2$
- **Kinetic Energy**: the mass $E_{\text{kin.}} = \frac{1}{2}\,m\,v(t)^2$

The total energy is:

$$E_{\text{total}} = E_{\text{pot.}} + E_{\text{kin.}} = \frac{1}{2}\,k\,x(t)^2 + \frac{1}{2}\,m\,v(t)^2$$

$$\text{Potential Energy :} \qquad E_{\text{pot.}} = \frac{1}{2}\,k\,x(t)^2 = \frac{1}{2}\,k\,A^2\,\cos^2(\omega t + \phi)$$

$$\text{Kinetic Energy :} \qquad E_{\text{kin.}} = \frac{1}{2}\,m\,v(t)^2 = \frac{1}{2}\,m\,A^2\,\omega^2\,\sin^2(\omega t + \phi)$$

$$\begin{aligned} E_{\text{total}} &= E_{\text{pot.}} + E_{\text{kin.}} \\ &= \frac{1}{2} A^2 \left(k \cos^2(\omega t + \phi) + m\omega^2 \sin^2(\omega t + \phi) \right) \end{aligned}$$

Substitute $\omega = \sqrt{\frac{k}{m}}$ or $k = m\,\omega^2$:

$$\begin{aligned} E_{\text{total}} &= \frac{1}{2} A^2 k \left(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right) = \frac{1}{2} k A^2 \\ E_{\text{total}} &= \frac{1}{2} m \omega^2 A^2 \left(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right) = \frac{1}{2} m \omega^2 A^2 = \frac{1}{2} m v_{\text{max.}}^2 \end{aligned}$$

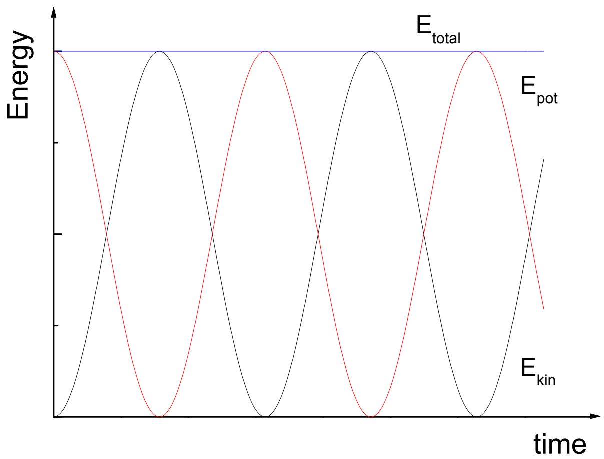

Figure 4.3: Potential energy and kinetic energy of the oscillation as a function of time.

Velocity at a given position:

$$\begin{aligned} E_{\text{total}} &= E_{\text{pot.}} + E_{\text{kin.}} \\ \frac{1}{2} k A^2 &= \frac{1}{2} m v^2 + \frac{1}{2} k x^2 \\ m v^2 &= k (A^2 - x^2) \\ v &= \pm \sqrt{\frac{k}{m}(A^2 - x^2)} = \pm\, \omega \sqrt{(A^2 - x^2)} \end{aligned}$$

4.1.2 Simple Harmonic Motion (SHM) of a Circular Motion

Rotation on a circle:

The position of a point on a circle is $P(x, y)$:

$$\begin{aligned} P_x(t) &= A \cdot \cos(\Theta) = A \cdot \cos(\omega t + \phi) \\ P_y(t) &= A \cdot \sin(\Theta) = A \cdot \sin(\omega t + \phi) \end{aligned}$$

where $\Theta = \omega t + \phi$.

Note: the side view of the rotation of a circular disk looks like a linear harmonic oscillation.

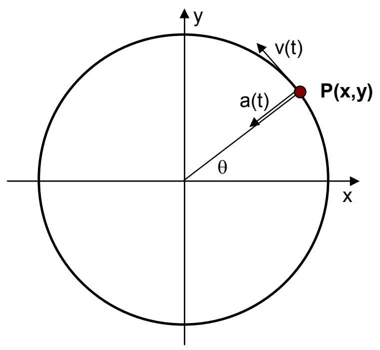

Figure 4.4: Circular motion of a point on a disk.

Velocity:

$$\begin{aligned} \vec{v}(t) &= \vec{\omega} \times \vec{x}(t) = \frac{dP(x, y)}{dt} \\ v_x(t) &= -\omega\, A\, \sin(\omega t + \phi) \\ v_y(t) &= \omega\, A\, \cos(\omega t + \phi) \end{aligned}$$

The angular frequency $\vec{\omega}$ is also a vector. Its direction is pointing along the axis of the rotation, i.e. along the z-direction and its magnitude is $\omega = \frac{2\pi}{T}$.

The vector of the velocity is tangential to the circle in forward direction of the motion.

Acceleration

$$\begin{aligned} \vec{a}(t) &= \vec{\omega} \times \vec{v}(t) = \vec{\omega} \times \vec{\omega} \times \vec{x}(t) = -\omega^2\, \vec{x}(t) \\ &= \frac{d^2 P(x, y)}{dt^2} \\ a_x(t) &= -\omega^2\, A\, \cos(\omega t + \phi) \\ a_y(t) &= -\omega^2\, A\, \sin(\omega t + \phi) \end{aligned}$$

The acceleration is parallel to the direction of the position of the point at $\vec{x}(t)$ but is pointing in the opposite direction, i.e. inwards to the centre of the circle: $\vec{a}(t) = -\omega^2\, \vec{x}(t)$.

4.1.3 The Pendulum

A mass is hanging on a string. After being moved to the side, the mass is swinging around the equilibrium position. The gravitational force is pulling it down. Note that the gravitational force is always acting in the direction of the motion.

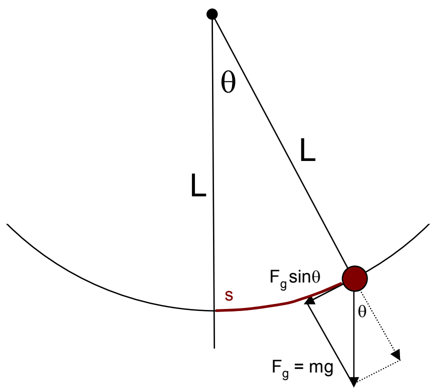

Figure 4.5: Gravitational force acing on the mass of a pendulum.

The corresponding gravitational force acting on the mass of the pendulum is:

$$F \quad = \quad = m\, g\, \sin \Theta$$

The equation of motion is:

$$m \cdot \frac{d^2 s(t)}{dt^2} + m\, g\, \sin \Theta(t) \quad = \quad 0$$

where $s(t)$ is the displacement to the side:

For small angles of Θ the following approximation can be made:

$$\sin \Theta \quad \approx \quad \tan \Theta \approx \Theta$$

$$\text{therefore :} \qquad \tan \Theta \quad \approx \quad \frac{s}{L} \qquad\qquad s \approx L \cdot \tan \Theta$$

The equation of motion can be written as a function of the angle Θ:

$$m\, L \cdot \frac{d^2 \Theta(t)}{dt^2} + m\, g\, \Theta(t) \quad = \quad 0$$

or:

$$\boxed{\frac{d^2 \Theta(t)}{dt^2} + \frac{g}{L}\, \Theta(t) \quad = \quad 0}$$

Note that the equation of motion does not contain the mass m. The only parameters which determine the oscillation of a pendulum are the gravitational constant $g = 9.81\, m/s^2$ and the length of the string L.

For the oscillation the following function can be assumed:

$$\Theta(t) = \Theta_0 \cdot \cos(\omega\, t + \phi)$$
$$\frac{d\Theta(t)}{dt} = -\Theta_0\, \omega \cdot \sin(\omega\, t + \phi)$$
$$\frac{d^2\Theta(t)}{dt^2} = -\Theta_0\, \omega^2 \cdot \cos(\omega\, t + \phi)$$

When inserted into the equation of motion one obtains:

$$\omega^2 = \frac{g}{L} \qquad \omega = \sqrt{\frac{g}{L}}$$

The Physical Pendulum

A physical pendulum is an object swinging about a pivot point that is not the center of mass (CM) of the object.

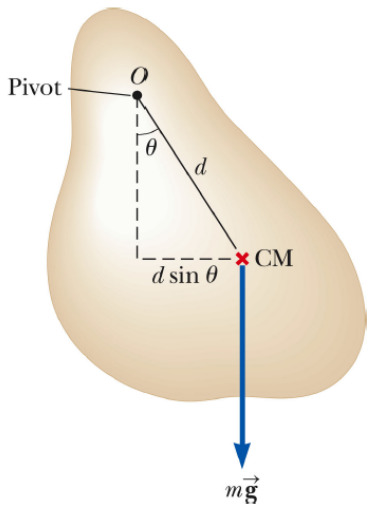

Figure 4.6: Physical pendulum.

$$\tau = \vec{r} \times \vec{F} = d\, m\, g \cdot \sin\Theta$$
$$\sum \tau_{\text{ext.}} = I \cdot a = I \cdot \ddot{\Theta}$$

$$-d\, m\, g \cdot \sin\Theta = I \cdot \frac{d^2\Theta}{dt^2}$$

$$\frac{d^2\Theta}{dt^2} = -\frac{d\, m\, g}{I} \cdot \Theta = -\omega^2\, \Theta$$

$$\omega = \sqrt{\frac{d\, m\, g}{I}} \qquad T = 2\pi\, \omega = \sqrt{\frac{I}{d\, m\, g}}$$

where I is the momentum of inertia of the object.

4.1.4 Damped Oscillation

In a real experiment there exists friction and air resistance, i.e. non-conservative forces are also acting on the movement of the mass and are slowing it down.

The friction acts against the motion of the mass. It is a ‘retarding force’.

The retarding force is usually proportional to the velocity:

$$F_r = -b \cdot v(t)$$

The equation of motion is then:

$$m \cdot a(t) = -b \cdot v(t) - k \cdot x(t)$$

$$\boxed{m \cdot \frac{d^2x(t)}{dt^2} + b \cdot \frac{d\,x(t)}{dt} + k \cdot x(t) = 0}$$

In this case it is an advantage if we use complex numbers to describe the oscillation.

Note: a complex number can be expressed as:

$$z = r\,(\cos\phi + i\sin\phi) = r \cdot e^{i\phi} \qquad \text{with}: \quad i^2 = -1$$

The function $x(t)$ of the oscillation is:

$$x(t) = A \cdot \cos(\omega t + \phi) = A \cdot e^{i(\omega t + \phi)}$$

$$\frac{dx(t)}{dt} = A \cdot (i\omega) \cdot e^{i(\omega t + \phi)}$$

$$\frac{d^2x(t)}{dt^2} = A \cdot (i\omega)^2 \cdot e^{i(\omega t + \phi)} = -A \cdot \omega^2 \cdot e^{i(\omega t + \phi)}$$

Inserted into the equation of motion gives:

$$-m\,A\,\omega^2 \cdot e^{i(\omega t + \phi)} + b\,A\,(i\omega) \cdot e^{i(\omega t + \phi)} + k\,A \cdot e^{i(\omega t + \phi)} = 0$$

$$-m\,A\,\omega^2 + b\,A\,(i\,\omega) + k\,A = 0$$

$$m\,\omega^2 - i\,b\,\omega - k = 0$$

$$\omega = i\frac{b}{2m} \pm \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$$

Note: $\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of the undamped harmonic oscillator.

Using this result for the angular frequency, the function of the oscillation can be written as:

$$x(t) = A \cdot \underbrace{\exp\left(-\tfrac{b}{2m}\,t\right)}_{\text{exp.decay}} \cdot \underbrace{\exp\left(i(\omega t + \phi)\right)}_{\text{oscillation}}$$

$$\text{where} \quad \omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

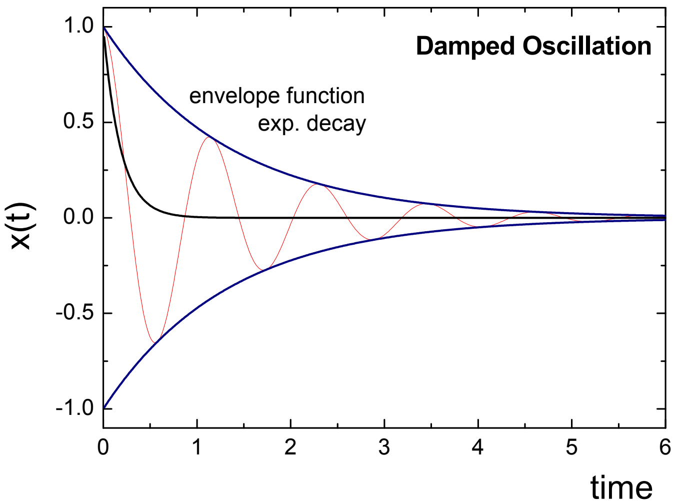

Figure 4.7: Damped oscillation. The envelope function of the oscillation is an exponential decay. The black line represents the case of the critical damping.

Special Cases:

Case I: Critical Damping

If the damping is increased, the oscillation stops at a certain **critical damping**:

$$\text{critical damping at :} \quad \omega = \sqrt{\tfrac{k}{m} - \left(\tfrac{b}{2m}\right)^2} = 0$$

$$\text{i.e.} \quad \tfrac{b}{2m} = \sqrt{\tfrac{k}{m}} = \omega_0$$

$$x(t) = A \cdot \exp\left(-\left(\tfrac{b}{2m}\,t\right)\right) = A \cdot \exp(-\omega_0\,t)$$

Case II: Overdamped

If the damping is larger than the critical damping, the oscillation stops completely and an exponentially decaying function is obtained:

$$\omega = i\,\tfrac{b}{2m} + \sqrt{-\left(\left(\tfrac{b}{2m}\right)^2 - \tfrac{k}{m}\right)} = i\,\tfrac{b}{2m} + \sqrt{i^2\left(\left(\tfrac{b}{2m}\right)^2 - \tfrac{k}{m}\right)}$$

$$\omega = i\left(\tfrac{b}{2m} + \sqrt{\left(\tfrac{b}{2m}\right)^2 - \tfrac{k}{m}}\right)$$

$$x(t) = A \cdot \exp\left(-\left(\tfrac{b}{2m} + \sqrt{\left(\tfrac{b}{2m}\right)^2 - \tfrac{k}{m}}\right)t\right)$$

4.1.5 Forced Oscillation

The damped harmonic oscillator is now attached to a motor which undergoes a periodic motion with a fixed angular frequency ω. The motor constantly adds energy to the system. The force of the motor acting on the mass is:

$$F(t) = F_0 \cdot \sin(\omega\, t)$$

The equation of motion is:

$$\boxed{m \cdot \frac{d^2x(t)}{dt^2} + b \cdot \frac{dx(t)}{dt} + k \cdot x(t) = F_0 \cdot \sin(\omega\, t)}$$

After a short time the mass oscillates with the same angular frequency as the motor. The motion of the mass can therefore described with the equation:

$$x(t) = A \cdot e^{i(\omega t + \phi)}$$

Inserted into the equation of motion gives:

$$-m\, A\, \omega^2 \cdot e^{i(\omega t+\phi)} b\, A\, (i\omega) \cdot e^{i(\omega t+\phi)} + k\, A \cdot e^{i(\omega t+\phi)} = F_0 \cdot e^{i\,\omega\, t}$$

1. Amplitude

$$-m\, A\, \omega^2 + b\, A\, (i\, \omega) + k\, A = F_0 \cdot e^{-i\phi}$$

$$A = \frac{F_0}{-m\, \omega^2 + i\, b\, \omega + k} \cdot e^{-i\phi}$$

$$= \frac{F_0/m}{(k/m) - \omega^2 + i\, b\, \omega/m} \cdot e^{-i\phi}$$

The amplitude is a real number and can be calculated using $|A| = \sqrt{A\, A^*}$

$$\boxed{|A| = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + (\frac{b\, \omega}{m})^2}}}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency of the undamped harmonic oscillator, i.e. its eigenfrequency.

2. Phase Shift

The phase shift ϕ is the angular difference between the motor and the mass.

It can be calculated by using the following equation for complex numbers: $\tan\phi = \frac{Im(A)}{Re(A)}$.

$$\tan\phi = \frac{b\, \omega/m}{\omega_0^2 - \omega^2}$$

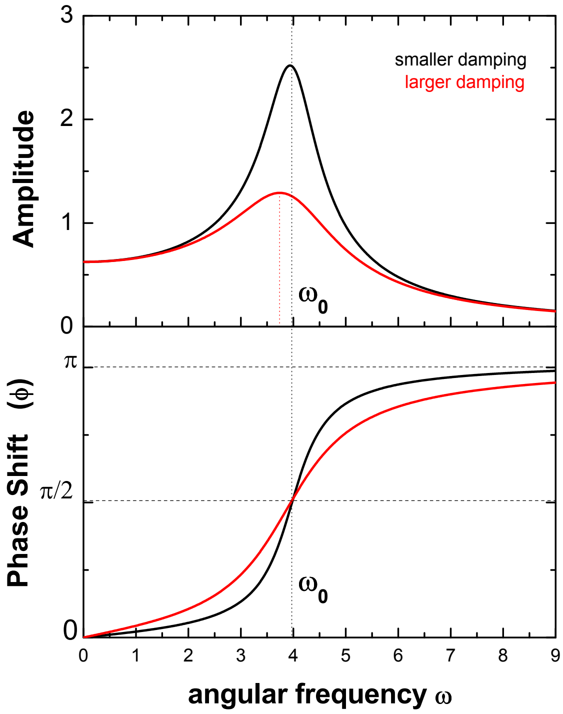

Figure 4.8: Amplitude $|A|$ and phase shift ϕ of the forced oscillation as a function of the angular frequency of the motor.

Case I: small angular frequency

The mass follows the motor instantaneously. The phase shift between the motor and the mass is zero ($\phi = 0$), i.e. both move 'parallel'. The amplitude is identical to the amplitude of the motor.

Case II: Resonance Catastrophe

Close to the eigenfrequency of the oscillator, i.e. at $\omega \approx \omega_0$, a resonance can appear. Depending on the size of the damping, the amplitude of the oscillation can reach extremely large values with even catastrophic results (see e.g. the collapse of the Tacoma Narrows Bridge in 1940).

The maximum of the resonance occurs at a frequency of:

$$\omega_{\text{res.}} = \frac{\omega_0}{(1-(\omega/\omega_0'))^4}$$

where ω is the frequency of the motor and ω_0' is the eigenfrequency of the damped harmonic oscillator.

At the resonance frequency the phase shift has a value of $\phi = \pi/2$, i.e. in every moment when the the mass is moving up, the motor is pulling it up. The same holds for the down movement. Therefore, a maximum of energy is transferred to the harmonic oscillator.

Without damping $b = 0$ the amplitude converges towards infinity, i.e. a resonance catastrophe is unavoidable. For large damping the maximum amplitude decreases significantly and a resonance catastrophe can be avoided.

Case III: large angular frequency

At very large frequencies of the motor ($\omega \gg \omega_0$) the mass of the oscillator can not follow any more. The phase shift reaches a value of $\phi = \pi$ (movement in opposite direction)) and the amplitude of the oscillator converges to zero.

Energy of the forced oscillation

Energy:

$$\begin{aligned} E_{\text{kin}} &= \tfrac{1}{2} m\, v^2 \;=\; \tfrac{1}{2} m\, \omega^2\, x_0^2\, \cos^2(\omega T + \phi) \\ \langle E_{\text{kin}} \rangle_{\text{avg.}} &= \tfrac{1}{4} m\, \omega^2\, x_0^2 \\ &= \tfrac{1}{4} m\, \omega^2\, A^2 \end{aligned}$$

Note that the following holds for the average value: $\cos^2(\phi) = \frac{1}{2}$.

The energy at the maximum is:

$$E_{\text{kin}} = \tfrac{1}{2} m\, \omega^2\, A^2$$

The potential energy is:

$$\begin{aligned} E_{\text{pot}} &= \tfrac{1}{2}\, k\, x^2 \;=\; \tfrac{1}{2} k\, x_0^2\, \sin^2(\omega T + \phi) \\ &= \tfrac{1}{2} m\, \omega_0^2\, x_0^2\, \left(1 - \cos^2(\omega T + \phi)\right) \end{aligned}$$

Therefore the total energy of the forced harmonic oscillator is:

$$\begin{aligned} E_{\text{total}} &= E_{\text{kin}} \;+\; E_{\text{pot}} \\ &= \tfrac{1}{2} m\, x_0^2\, \left(\omega_0^2 + (\omega^2 - \omega_0^2)\, \cos^2(\omega t + \phi)\right) \end{aligned}$$